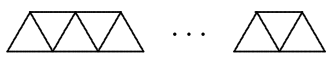
Singapore Mathematical Society

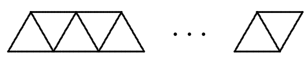
Singapore Mathematical Olympiad (SMO) 2018

(Senior Section, Round 2 solutions)

1. (*See Junior Section Round 2 Question 5 for the earlier part.*) To complete the proof, we now show that n cannot be 3 or 4. First note that when a triangle and a square meet at the boundary, they meet at a vertex of the polygon and the angle at the vertex is 150°. Thus if there are both triangles and squares at the boundary, there are two 150° angles. This is not possible. Thus the boundary of the polygon formed consists entirely of triangles or squares. Therefore there are four possibilities: an equilateral triangle, a parallelogram , a trapezium both with angles 60°, 60°, 120°, 120°, and a rectangle. For the rectangle, the boundary consists of squares. Removing the squares from the bottom layer either leaves nothing or another rectangle. This means that no triangle is used, a contradiction. For the first three cases, the boundary consists of only triangles. The bottom level consists of triangles arranged as shown in the figure below. Removing these triangles leaves a smaller figure of the same shape. We conclude that no square is used. Therefore n cannot be 3 or 4.



or



Alternatively, as before, there are four possible polygons. For the rectangle, the area A is an integer. The other three all have integral side lengths. Since the interior angles are either 60° or 120°, the height is a rational multiple of $\sqrt{3}$. Hence the area is of the form $A = q\sqrt{3}$ where $q \in \mathbb{Q}$. If m triangles and n squares are used, then $A = n + m\sqrt{3}/4$. If A is an integer, $m = 0$, and if $A = q\sqrt{3}$, $n = 0$. Thus only triangles or squares are used.