Singapore International Mathematical Olympiad 2008
Senior Team Training

Take Home Test

Submit written solutions by 12 April 2008.

(1) Show that the equation $15x^2 - 7y^2 = 9$ has no solution in integers.

(2) Let n and k be positive integers. Prove that

$$(n^4-1)(n^3-n^2+n-1)^k + (n+1)n^{4k-1}$$

is divisible by n^5+1.

(3) Let

$$\begin{aligned} f(x) &= (x+1)^p(x-3)^q \\ &= x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n, \end{aligned}$$

where p and q are positive integers.
(a) Given that $a_1 = a_2$, prove that $3n$ is a perfect square.
(b) Prove that there exist infinitely many pairs (p, q) of positive integers p and q such that the equality $a_1 = a_2$ is valid for the polynomial $p(x)$.

(4) Show that if $m < n$, then $2^{2^m}+1$ divides $2^{2^n}-1$. Hence deduce that $2^{2^m}+1$ and $2^{2^n}+1$ are relatively prime. Conclude that there are infinitely many primes.

(5) Let x, y, z be positive numbers so that $xyz = 1$. Prove that

$$x+y+z \geq \sqrt[3]{\frac{z}{x}} + \sqrt[3]{\frac{x}{y}} + \sqrt[3]{\frac{y}{z}}.$$

(6) Let $p_1, p_2, \ldots, p_n$ $(n \geq 2)$ be any rearrangement of $1, 2, \ldots n$. Show that

$$\frac{1}{p_1+p_2} + \frac{1}{p_2+p_3} + \cdots + \frac{1}{p_{n-1}+p_n} > \frac{n-1}{n+2}.$$

(7) From a point P outside a circle, tangent lines PA and PB are drawn with A and B on the circle. A third line PCD meets the circle at C and D, with C lying in between P and D. A point Q is chosen on the chord CD so that $\angle DAQ = \angle PBC$. Show that $\angle DBQ = \angle PAC$.

(8) In triangle ABC, $\angle A = 60^\circ$ and $AB > AC$. The altitudes BE and CF intersect at H. Points M and N are chosen on the segments BH and HF so that $BM = CN$. If O is the circumcircle of ABC, find the ratio

$$\frac{MH + NH}{OH}.$$

(9) On the plane, there are 3 mutually and externally disjoint circles Γ_1, Γ_2 and Γ_3 centred at X_1, X_2 and X_3 respectively. The two internal common tangents of Γ_2 and Γ_3, (Γ_3 and Γ_1, Γ_1 and Γ_2) meet at P, (Q, R respectively). Prove that X_1P, X_2Q and X_3R are concurrent.

(10) The excircle centred at I_a with respect to $\angle A$ of $\triangle ABC$ touches the sides AB, BC and AC or their extensions at E, D and F respectively. Let H be the foot of the perpendicular from B onto I_aC. Prove that E, H, F are collinear.