31st China Mathematical Olympiad

Jiangxi Yingtan

Day 1

2015.12.16 8:00 ~12:30

1. Let $a_1, a_2, \cdots, a_{31}, b_1, b_2, \cdots, b_{31}$ be positive integers satisfying

(i) $a_1 < a_2 < \cdots < a_{31} \leq 2015$, $b_1 < b_2 < \cdots < b_{31} \leq 2015$; and

(ii) $a_1 + a_2 + \cdots + a_{31} = b_1 + b_2 + \cdots + b_{31}$.

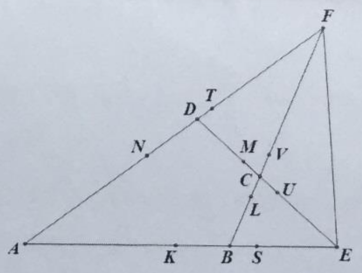
Find the maximum value of the sum $S = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{31} - b_{31}|$.

2. As shown in the figure, $ABCD$ is a convex quadrilateral. Points K, L, M, N lie on the sides AB, BC, CD, DA respectively, such that

$$\frac{AK}{KB} = \frac{DA}{BC}, \quad \frac{BL}{LC} = \frac{AB}{CD}, \quad \frac{CM}{MD} = \frac{BC}{DA}, \quad \frac{DN}{NA} = \frac{CD}{AB}.$$

The extensions of AB and DC meet at point E, and the extensions of AD and BC meet at point F. The inscribed circle of triangle AEF touches the sides AE, AF at points S, T respectively; the inscribed circle of triangle CEF touches the sides CE, CF at points U, V respectively.

Prove that if points K, L, M, N are cyclic, then points S, T, U, V are cyclic.



3. Let p be an odd prime, and $a_1, a_2, \cdots, a_p$ be integers. Prove that the following two statements are equivalent:

(I) There exists a polynomial $f(x)$ with integer coefficients and of degree less than or equal to $\frac{p-1}{2}$, such that $f(i) \equiv a_i \pmod{p}$ holds for every positive integer $i \leq p$.

(II) For each positive integer $d \leq \frac{p-1}{2}$, the congruence equality

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod{p}$$

holds, where the indices are considered modulo p, that is $a_{p+n} = a_n$.

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Day 2

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4. Let $n \geq 3$ be an integer, and k be the number of primes not exceeding n. Let A be a subset of $\{2, 3, \cdots, n\}$ whose cardinality is less than k, satisfying that any element of A is not a multiple of any other element of A. Prove that there exists a k-element subset B of $\{2, 3, \cdots, n\}$, such that any element of B is not a multiple of any other element of B, and B contains A.

5. Let $ABCD$ be an arbitrary convex quadrilateral in the plane. Prove that there exists a square $A'B'C'D'$ (the vertices may be ordered clockwisely or counterclockwisely as you want), such that $A' \neq A$, $B' \neq B$, $C' \neq C$, $D' \neq D$, and the lines AA', BB', CC', DD' pass through a common point.

6. 100 players participate in a tournament. Any two players x and y play exactly once and there is no draw in this game. We use $x \to y$ to mean that x beats y. The tournament is called *friendly* if for each pair of players x, y, there exists a sequence of players $u_1, u_2, \cdots, u_k$ $(k \geq 2)$, such that $x = u_1 \to u_2 \to \cdots \to u_k = y$.

(a) For any friendly tournament, prove that there exists a positive integer m, such that for each pair of players x, y, there exists a sequence of players $z_1, z_2, \cdots, z_m$ of length m (repetition is allowed in $z_1, z_2, \cdots, z_m$), satisfying $x = z_1 \to z_2 \to \cdots \to z_m = y$.

(b) For a friendly tournament T, let $m(T)$ denote the minimal value of m as stated in (a). Find the smallest possible value of $m(T)$.