2015 China Western Mathematical Invitation

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Second Day 17^{th} August, 8:00 am ~12:00 noon

Each problem is 15 marks

5. Let $ABCD$ be a convex quadrilateral with area S, and $AB = a, BC = b, CD = c,$ $DA = d$. For any permutation x, y, z, w of a, b, c, d, prove that $S \le \frac{1}{2}(xy + zw)$.

6. For a sequence $a_1, a_2, \cdots, a_m$ of real numbers, define the following sets

$$A = \{a_i \mid 1 \le i \le m\} \text{ and } B = \{a_i + 2a_j \mid 1 \le i, j \le m, i \ne j\}.$$

Let n be a given integer, and $n > 2$. For any strictly increasing arithmetic sequence $a_1, a_2, \cdots, a_n$ of integers, determine, with proof, the minimum number of elements of set $A \Delta B$, where $A \Delta B = (A \cup B) \setminus (A \cap B)$.

7. Let $a \in (0,1), f(z) = z^2 - z + a, z \in \mathbb{C}$. Prove the following statement holds:

For any complex number z with $|z| \ge 1$, there exists a complex number z_0 with $|z_0| = 1$, such that $|f(z_0)| \le |f(z)|$.

8. Let k be a positive integer, and $n = (2^k)!$. Prove that $\sigma(n)$ has at least a prime divisor larger than 2^k, where $\sigma(n)$ is the sum of all positive divisors of n.