33th International Mathematical Olympiad

Russia, July 1992.

1. Find all integers a, b, c with $1 < a < b < c$ such that $(a-1)(b-1(c-1)$ is a divisor of $abc-1$.

Soln. Write $x = a-1$, $y = b-1$ and $z = c-1$. The problem is equivalent to: Find all integers x, y, z with $0 < x < y < z$ such that xyz is a divisor of $xyz+xy+yz+xz+x+y+z$. Let $R(x,y,z) = (xyz+xy+yz+xz+x+y+z)/(xyz)$. Since $R(x,y,z)$ is an integer, x, y, z are either all even or all odd. Also

$$R(x,y,z) = 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz}.$$

If $x \geq 3$, then

$$1 < R(x,y,z) \leq R(3,5,7) < 2$$

which is impossible. If $x = 2$, then

$$1 < R(x,y,z) \leq R(2,4,6) < 3$$

and so $R(2,y,z) = 2$. This implies

$$(y-3)(z-3) = 11$$

which gives $y = 4, z = 14$.

If $x = 1$, then

$$1 < R(x,y,z) \leq R(1,3,5) < 4.$$

If $R(1,x,y) = 2$, then $2y+2z+1 = 0$ which is impossible. If $R(1,x,y) = 3$, then

$$(y-2)(z-2) = 5.$$

Thus $y = 3, z = 7$. So there are two possible solutions

$$(a,b,c) = (3,5,15), (2,4,8).$$

It is easy to check that both are indeed solutions.

2. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = y + f(x)^2 \quad \text{for all} \quad x, y \in \mathbb{R}. \tag{1}$$

Soln. Let $f(0) = s$ and put $x = 0$ in (1),

$$f(f(y)) = y + s^2 \quad \text{for all} \quad y \in \mathbb{R}. \tag{2}$$

Put $y = 0$ in (1),

$$f(x^2 + s) = f(x)^2 \quad \text{for all} \quad x \in \mathbb{R}. \tag{3}$$

Put $x = 0$ in (3),

$$f(s) = s^2 \tag{4}$$

Add (3) and (4),

$$s^2 + f(x^2 + s) = f(x)^2 + f(s) \quad \text{for all} \quad y \in \mathbb{R}. \tag{5}$$

Apply f to both sides and use (2), (3), (4),

$$x^2 + s + s^4 = s + (x + s^2)^2 \quad \text{for all} \quad x \in \mathbb{R}$$

This gives $f(0) = s = 0$. Thus, from (2), (3),

$$f(f(x)) = x, \quad f(x^2) = f(x)^2 \quad \text{for all} \quad x \in \mathbb{R}. \tag{6}$$

The latter implies that $f(x) \geq 0$ if $x \geq 0$. If $f(x) = 0$ for some $x \geq 0$, then

$$0 = f(x)^2 = f(x^2) = f(x^2 + f(x)) = x + f(x)^2 = x.$$

Thus

$$f(x) > 0 \quad \text{for all} \quad x > 0. \tag{7}$$

Replace y by $f)y)$ and x by $\sqrt{x}$, we have

$$f(x + y) = f(x) + f(y) \quad \text{for all} \quad x > 0, y \in \mathbb{R}. \tag{8}$$

Now if $x > y$, then

$$f(x) = f((x - y) + y) = f(x - y) + f(y) > f(y).$$

Suppose there exists x such that $f(x) > x$, then $x = f(f(x)) > f(x)$, a contradiction. Suppose there exists y such that $f(y) < y$, then $y = f(f(y)) < f(y)$, again impossible. Thus $f(x) = x$ for all x. This is indeed a solution.

3. Consider nine points in sapce, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either coloured blue or red or left uncoloured. Find the smallest value of n such that whenever exactly n edges are coloured the set of coloured edges necessarily contains a triangle all of whose edges have the same colour.

Soln. If three edges are not coloured, then the three edges are either idependent, form a triaangle, form a path or form a star. In each case, the coloured edges contained a K_6. Thus there is a monochromatic triangle. If there are uncloured edges, then the following example shows that it is possibld that there are no monochromatic triangles. Label the vertices

a_i, b_i, $i = 1, 2, 3, 4$ and x. Leave the edges $a_1a_3, a_2a_4, b_1b_3, b_2b_4$ uncoloured. Coloured the following edges red:

$$
\begin{gathered}
xa_i, \quad i = 1, 2, 3, 4 \\
b_1b_3, b_1b_4, b_2b_3, b_2b_4, \\
a_1b_1, a_1b_2, a_2b_1, a_2b_2, \\
a_3b_3, a_3b_4, a_4b_3, a_4b_4.
\end{gathered}
$$

The remaining edges are coloured blue.

4. In the plane let C be a circle, L a line tangent to the circle and M a point on L. Find the locus of all points P with the folloiwng property:

> there exist two points Q, R on L such that M is hte midpoint of QR and C is the incribed circle of triangle PQR.

Soln. Consider any triangle ABC with incircle Γ and excircle Γ' which touches the side BC externally. If X and Y are the points of where BC touches the incircle and excircle, respecitvely. We have the following:

(1) $CY = BX = s - AC$, where s is the semiperimater of ABC. This follows from the fact that $AB + BY = AC + CY$ and $AB + BY + AC + CX = 2s$. (2) It follows from (1) that the midpoint of BC and XY are the same.

(2) If XZ is a diameter of the incircle, then the homothety with centre A that takes the incircle to the excircle takes Z to Y.

Now we solve the problem: Let X be the point where L touches the circle C and XZ be a diameter of C. Also let Y be the point on L which is symmetric to X with respect to M. We claim that the locus is the open ray on the line YZ emanating away from the circle.

If P is point with the desired property, then the homothety with centre at P taking the incircle C of PQR to its excircle takes Z to Y. Thus P lies on the open ray.

Conversely, any point on the open ray has the desired property.

5. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane respectively. Prove that

$$|S|^2 \leq |S_x||S_y||S_z|$$

where $|A|$ denotes the number of elements in the finite set set A. (Note: the orthogonal projection of a point onto a plane is the foot of the perpendicular from the point to the plane).

Soln. For each (i, j) let S_{ij} be the set of points of the type (x, i, j), i.e., the set of points that project to (i, j). Then

$$S = \cup_{(i,j) \in S_x} S_{ij}.$$

By Cauchy's inequality,

$$|S_x| \sum_{(i,j)\in S_x} |S_{ij}|^2 \geq |S|^2.$$

Let $X = \cup_{(i,j)\in S_x} S_{ij} \times S_{ij}$. Then $|X| = \sum_{(i,j)\in S_x} |S_{ij}|^2$. The map $f : X \to S_y \times S_z$ defined by $f((x,i,j),(x',i,j)) = ((x,i),(x,j))$ is certainly injective. So $|X| \leq |S_y||S_z|$.

Second soln. First we assume that all the points of S lie on a plane prallel to the xy-plane. In this case, we have $|S_x||S_y| \geq |S|^2$. But $|S_z| \geq 1$ so the result holds.

So we assume that the result holds for the case where the points of S lie on at most n different planes parallel to the xy-plane. Consider the case where the points lie on $n+1$ different planes. Find a plane, parallel to the xy-plane, which divides the points of S into two nonempty parts T, U, but itself does not contain any points of S. Then $|S| = |U| + |T|$, $|U_x| + |T_x| = S_x$ and $|U_y| + |T_y| = |S_y|$ and $|U_z|, |T_z| \leq |S_z|$. By the induction hypothesis, we have

$$\begin{aligned}
|S| &= |U| + |T| \\
&\leq (|U_x||U_y||U_z|)^{1/2} + (|T_x||T_y||T_z|)^{1/2} \\
&\leq |S_z|^{1/2}((|U_x||U_y|)^{1/2} + (|T_x||T_y|)^{1/2}) \\
&\leq |S_z|^{1/2}(|U_x| + |T_x|)^{1/2}(|U_y| + |T_y|)^{1/2} \\
&= (|S_x||S_y||S_z|)^{1/2}
\end{aligned}$$

6. For each positive integer n, $S(n)$ is defnied to be the greatest integersuch that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive square integers.

(a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.

(b) Find an integer n such that $S(n) = n^2 - 14$.

(c) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

Soln. (a) From $n^2 = 1 + 1 + \cdots + 1$, we see that n^2 can be written as a sum of n^2 squares. We can combine 4 ones to get 2^2, 9 ones to get 3^2, etc, to reduce the number of squares by 3, 6, 8, 9, 11, 12, but bot 13. Thus $s(n) \leq n^2 - 14$.

(b) We'll show that 169 can written as a sum of t squares, $t = 1, 2, \ldots, 155$.

First $169 = 9 + 4 + 4 + 152 \times 1$ is a sum of 155 squares. By grouping 4 ones into a 4, we can get $t = 155, 152, \ldots, 41$. By group 4 fours into 16, we can get $t = 38, 35, \ldots, 11$. By group 4 sixteens into 64, we get $t = 8, 5$. Of course $t = 2$ is obtained by $5^2 + 12^2$.

Next we start with $169 = 5 \times 4 + 149 \times 1$ as a sum of 154 squares. Group as before we get $t = 154, 151, \ldots, 7$. For $t = 4$, we can use $5^2 + 4^2 + 8^2 + 8^2$.

Next we start with $169 = 9 + 9 + 151 \times 1$ as a sum of 153 squares. Group as before we get $t = 153, 150, \ldots, 9$. Next we use $169 = 3^2 + 4^2 + 12^2 = 4 \times 2^2 + 2^2 + 12^2$ TO GET $t = 3, 6$. The list is now complete.

Alt: One such n is 13. (In fact this is the smallest as it it is the smallest numberthat can be written as the sum of 2 and 3 squares.)

$$169 = 13^2 = 5^2 + 12^2 = 3^2 + 4^2 + 12^2 = 5^2 + 4^2 + 8^2 + 8^2 = 3^2 + 4^2 + 4^2 + 8^2 + 8^2.$$

Using the fact that

$$(2r)^2 = r^2 + r^2 + r^2 + r^2 \qquad (*)$$

and

$$169 = 3^2 + 4^2 + 4^2 + 8^2 + 8^2$$

we can write 169 as a sum of $3t+2$, $1 \le t \le 53$. Replacing 3^2 by $2^2 + 2^2 + 1$ in the above, we can also write 169 as a sum of $3t+2$, $2 \le t \le 54$, squares. Using $169 = 3^2 + 4^2 + 12^2$ and $(*)$ again, we can write 169 as the sum of $3t$, $1 \le t \le 11$, squares. Moreover in the last sum, the summands consists of 16 ones and 17 nines. Now use the $3^2+3^2+3^2+1 = 7 \times 2^2$, we can write 169 as a sum of $3t$, $12 \le t \le 16$. In the final sum, the summnads consists of 11 ones, 35 fours and 2 nines. The fours and now be broken up using $(*)$ again to obtain sums of $3t$, $17 \le t \le 51$. The list is now complete.

(c) Let n be a number such that $S(n^2) = n^2 - 14$. We claim that $2n$ also has the property. If $n^2 = a^2 + b^2 + \cdots$, then $(2n)^2 = (2a)^2 + (2b)^2 + \cdots$. splitting each even squares as before, we see that $(2n)^2$ be written as a sum of t squares, $1 \le t \le (2n)^2 - 56$. Since $4n^2 > 169$, we can use the grouping described above to get all the representations from $4n^2 - 55$ to $4n^2 - 14$.