34th International Mathematical Olympiad

Turkey, July 1993.

1. Let $f(x) = x^n + 5x^{n-1} + 3$ where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two polynomials, each of which has all its coefficients integers and degree at at least 1.

Soln. Suppose on the contrary that $f(x) = g(x)h(x)$ where $g(x) = a_0 + a_1x + \cdots + a_sx^s$, $h(x) = b_0 + b_1x + \cdots + b_tx^t$, $s + t = n$, $a_s = b_t = 1$ and $s, t \geq 1$. Then $a_0b_0 = 3$. Thus 3 divides exactly one of a_0, b_0, say $3 \mid a_0$. Let k be the smallest integer such that $3 \nmid a_k$. Then, since $k \leq n-1$ the coeeficient of x^k in $f(x)$ is either 0 or 5. If it is 0, i.e., $k < n-1$, then

$$0 = a_0b_k + a_1b_{k-1} + \cdots + a_{k-1}b_1 + a_kb_0.$$

Since $3 \mid a_i$ for $i = 0, \ldots, k-1$, we have $3 \mid a_kb_0$. But $3 \nmid b_0$. So $3 \mid a_k$, a contradiction.

If it is 5, then $k = n-1$. Thus $h(x) = 1 + x$ and $a_i = 3a_i'$ for $i = 1, 2, \ldots, n-2$. Thus we have

$$f(x) = 3 + 3(a_1' + 1)x + 3(a_2' + a_1')x^2 + \cdots + 3(a_{n-2}' + a_{n-3}')x^{n-2} + (3a_{n-2}' + 1)x^{n-1} + x^n.$$

Thus $a_i' \equiv \pm 1 \pmod 5$ for $i = 1, 2, \ldots, n-2$. This means $5 \nmid 3a_{n-2}' + 1 = 5$, a contradiction.

2. Let D be a point inside the acute-angled triangle ABC such that

$$\angle ADB = \angle ACB + 90^\circ \quad \text{and} \quad AC \cdot BD = AD \cdot BC$$

(a) Calculate the value of the ratio $\frac{AB \cdot CD}{AC \cdot BD}$

(b) Prove that the tangent at C to the circumcircles of the triangles ACD and BCD are perpendicular.

Soln. Draw DE equal and perpendicular to DB. Then $\angle ADE = \angle ACB$. Also

$$\frac{AD}{AC} = \frac{BD}{BC} = \frac{DE}{BC}.$$

If follows that triangles ADE and ACB are similar, since they have sides about equal angles being proportional. Hence $\angle CAB = \angle DAE$ and $AB/AE = C/AD$. Now

$$\angle CAD = \angle CAB - \angle DAB = \angle DAE - \angle DAB = \angle BAE.$$

Consequenctly triangles CAD and BAE are similar. Therefore

$$\frac{AC}{AB} = \frac{CD}{BE} = \frac{CD}{\sqrt{2}BD}$$

since DBE is a right-angled isosceles triangle. hence the value of the required ratio is $\sqrt{2}$.

Let CT, CU be the tangents at C to the circles ACD, BCD respectively. Then $\angle DCT = \angle DAC$ and $\angle DCU = \angle DBC$. Now $\angle ADE + \angle DAB + \angle DBA = 180^\circ - 90^\circ = 90^\circ$. So by similar triangles

$$\angle ACB + \angle CAB - \angle CAD + \angle ABC - \angle DBC = 90^\circ$$

and therefore $\angle CAD + \angle DBC = 180^\circ - 90^\circ = 90^\circ$. Thus $\angle TCU = 90^\circ$ as required.

3. On an infinite chessboard, a game is played as follows:

At the start, n^2 pieces are arranged on the chessboard in an $n \times n$ block of adjoining sqaures, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacnet occupied square to an an unoccupied square immediately beyond. The piece which has been jumped over is hten removed.

Find those values of n for which the game can end with one piece remaining on the board.

Soln. We replace the squares by $(x, y) \in \mathbb{Z}^2$ and assume the pieces are originaly in $1 \leq x, y \leq n$. For every $k \in \mathbb{Z}$, denote by $s(k, j)$ the number of occupied points satisfying $x + y = k$ after j moves and for $i = 0, 1, 2$, let

$$S_i(j) = \sum_{i \equiv k \pmod 3} s(k, j).$$

If $n = 3p$, then

$$S_0(0) = 2(3 + 6 + \cdots + 3p - 3) + 3p = 3p^2,$$

whence $S_1(0) = S_2(0) = 3p^2$. So either every $S_i(0)$ is even or every $S_i(0)$ is odd. Every move changes $s(k, j)$ for three consecutive values of k, two s's diminish by one and one increases by one. So every move reverses the parity of all the S_i's. If the game should end as required, one of the $S_i(n^2 - 1)'s$ should be one and the others zero. So if n is a multiple of 3, the game cannot end as required.

Now assume that n is not a multiple of 3. If $n = 2$, the moves

$$(1, 1) \to (3, 1), \quad (1, 2) \to (3, 2), \quad (3, 2) \to (3, 0)$$

show that the end situation can be achieved. If $n \geq 3$, the moves

$$(2, 1) \to (0, 1), \quad (1, 3) \to (1, 1), \quad (0, 1) \to (2, 1)$$

show that one can always remove the three pieces in a row from an L-shpaed 4-piece configurationm which is bordered on one long side by an unoccupied area. In this manner one can remove two strips from the original $n \times n$ square leaving an $(n-3) \times (n-3)$ square. The process can be continued until a 2×2 square is left. Thus the game will end.

4. For tthree points P, Q, R in the plane, we define $m(PQR)$ to be the minimum of the lengths of the altitudes of the triangle PQR (where $m(PQR) = 0$ when P, Q, R are collinear.) Let A, B, C be given points in the plane. Prove that for any point X in the plane,

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

Soln. Key idea is the $m(ABC) = 2[ABC]/BC$ if BC is the longest side.

5. Let $\mathbb{N} = \{1, 2, \ldots\}$. Determine whether or not there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 2$,

$$f(f(n) = f(n) + n, \quad f(n) < f(n+1) \quad \text{for all} \quad n \in \mathbb{N}.$$

Soln. Let $\alpha = (\sqrt{5} + 1)/2$. Since $\alpha^2 - \alpha - 1 = 0$, the function $g(x) = \alpha x$ satisfies

$$g(g(n)) - g(n) - n = 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

We shall that the function $f(n) = \lfloor g(n) + \frac{1}{2} \rfloor$ satisfies the requirements. We oberseve:

(i) f is strictly increasing, because $\alpha > 1$ so $g(n+1) > g(n) + 1$ holds.

(ii) Since $2 < \alpha + \frac{1}{2} < 3$ holds, then $f(1) = 2$.

(iii) By the definition, $|f(n) - g(n)| < 1/2$ holds for all n. Then $f(f(n)) = f(n) + n$ follows from the fact that $f(f(n)) - f(n) - n$ is an integer and the estimate:

$$\begin{aligned}
&|f(f(n)) - f(n) - n| \\
&\quad = |g(g(n)) - g(n) - g(g(n)) + f(f(n)) - f(n) + g(n)| \\
&\quad = |g(g(n)) - f(f(n)) + f(n) - g(n)| \\
&\quad = |g(g(n)) - g(f(n)) + g(f(n)) - f(f(n)) + f(n) - g(n)| \\
&\quad = |(\alpha - 1)(g(n)) - f(n)) + g(f(n)) - f(f(n))| \\
&\quad \leq (\alpha - 1)|(g(n)) - f(n)| + |g(f(n)) - f(f(n))| \\
&\quad \leq \frac{\alpha - 1}{2} + \frac{1}{2} < 1.
\end{aligned}$$

Note $f(n) = \lfloor \alpha n + \frac{1}{2} \rfloor$ also works.

6. Let $n > 1$ be an integer. there are n lamps $L_0, L_1, \ldots, L_{n-1}$ arranged in a circle. Each lamp is either ON or OFF. A sequence of steps $S_0, S_1, \ldots$ is carried out. Step S_j affects the state of L_j only (leaving the state of all other lamps unaltered) as follows:

if L_{j-1} is ON, S_j changes the state of L_j from ON to OFF ot from OFF to ON;

if L_{j-1} is OFF, S_j leaves the state of L_j unchaged.

Initially all lamps are ON. Show that

(a) there is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are ON again;

(b) if n has he form 2^k then all lamps are ON after $n^2 - 1$ steps;

(c) if n has the form $2^k + 1$ then all the lamps are ON after $n^2 - n + 1$ steps.

Soln. Let 1 represents ON and 0 represents OFF and we work in mod 2.

(a) The number of states is finite so after a finite number of steps there must be a repetition. Hence the operation is reversible, the first to repeat is the initial state where every lamp is ON.

(b) Let the number of lamps be $n = 2^k$. Let P_m be the state where the lamps are partitioned in blocks of 2^m, where lamps in the blocks $1, 3, 5, \ldots$ are all 1, while those in the other blocks are all 0. A round of operations is simply a consecutive set of 2^k operations starting with the first lamp. We claim that after 2^m rounds of operations, we get the state P_{m+1}. The proof is by induction on m. For $m = 0$ it is clear that after one round we can go from

$$1010101010\ldots \quad \text{to} \quad 1100110011\ldots.$$

So we assume that the result is true for m. Now consider the state P_{m+1}. After one round we get

$$(1010\ldots)(0000\ldots)(1010\ldots)(0000\ldots)\ldots$$

where the first parenthesis is a block of 2^{m+1} alternating between 1 and 0, the second is a block of 0 and so on. By the induction hypothesis, we can transformed the first block to a block of ones in $1 + 2 + \cdots + 2^m = 2^{m+1} - 1$ rounds. In the previous round, i.e., round $2^{m+1} - 2$, the first block consists of a one follow by zeroes while the second block is all zeroes. The same goes for the thrid and fourth blocks, etc. After another round, the first two blocks are all ones while the the next two are all zeroes, and so on. Thus after 2^{m+1} rounds P_m is transformed into P_{m+1}.

Now after $n - 1$ steps, we get P_0. Thus after $1 + 2 + \cdots + 2^{k-1} = 2^k - 1 = n - 1$ rounds, we get P_k. But P_k is the state where all lamps are ON. Thus the total number of steps is $(n - 1) + n(n - 1) = n^2 - 1$.

(c) After one round, we get

$$01010\cdots 10$$

which is a 0 followed by P_1. After one more round we get a 0 followed by P_2, and so on. So after a total of $1 + 1 + 2 + 4 + \cdots + 2^{k-1} = 2^k = n - 1$ rounds, we get a 0 followed by ones. One more step will turned the lamps into ON. Thus the total number of moves is $n(n - 1) + 1 = n^2 - n + 1$.

Second soln. (Official) Represent ON by 1 and OFF by 0 and work in mod 2. Suppose we are given the state

$$(a_0, a_1, \ldots, a_{n-1}) \qquad (*)$$

After S_0, we get $(a_0 + a_{n-1}, a_1, a_2, \ldots, a_{n-1})$. Due to rotational symmetry, this state is equivelent to

$$(a_1, a_2, \ldots, a_{n-1}, a_0 + a_{n-1}) \qquad (\dagger).$$

This is convenient as we shall then always apply the same operation. Thus an operation transforms $(*)$ into $(\dagger)$. It's convenient to represent the state $(*)$ by the polynomial

$$P(x) = a_{n-2} + a_{n-3}x + a_{n-4}x^2 + \cdots + a_0 x^{n-2} + a_{n-1}x^{n-1}.$$

A one step rotation can be conveniently represented by $xP(x)$. The state $(\dagger)$ is represented by the polynomial

$$Q(x) = a_{n-1} + a_{n-2}x + a_{n-3}x^2 + \cdots + a_1 x^{n-2} + (a_0 + a_{n-1})x^{n-1}.$$

Now

$$xP(x) - Q(x) = a_{n-1}x^n - a_{n-1}x^{n-1} - a_{n-1} = \begin{cases} x^n + x^{n-1} + 1 & \text{if } a_{n-1} = 1 \\ 0 & \text{if } a_{n-1} = 0 \end{cases}$$

Thus

$$xP(x) \equiv Q(x) \pmod{x^n + x^{n-1} + 1}$$

For (a), we need to show that there exists a positive integer $M(n)$ such that $x^{M(n)} \equiv 1$. Since the residue class is finite, there are two integers p, q such that $x^p \equiv x^{p+q}$. This implies $x^p(x^q - 1) \equiv 0$. Since $x^p \not\equiv 0$, we have $x^q \equiv 1$.

For (b), we take $n = 2^k$, and need to show $x^{n^2-1} \equiv 1$. We have

$$x^{n^2} \equiv (x^{n-1} + 1)^n \equiv x^{n^2-n} + 1.$$

Thus

$$(1 + x^n)x^{n^2-n} \equiv x^{n^2} \equiv 1.$$

For (c), we take $n = 2^k + 1$. We have

$$x^{n^2-1} \equiv (x^{n+1})^{n-1} \equiv (x + x^n)^{n-1} \equiv x(x^{n-1} + x^{n(n-1)}).$$

Thus

$$(1 + x^{n-1}x^{n^2-n} \equiv x^{n-1}.$$

Since $1 = x^{n-1} \equiv x^n$, we have $x^{n^2-n+1} \equiv 1$.