38th International Mathematical Olympiad

Argentina, July 1997.

1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along the edges of the squares. Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let $f(m,n) = |S_1 - S_2|$.

(a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

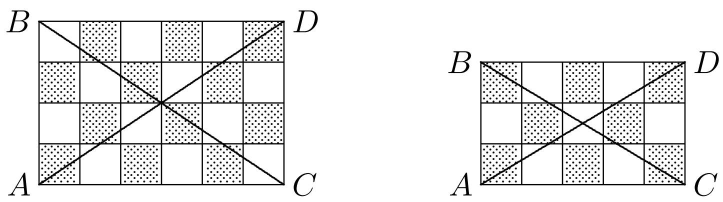
(b) Prove that $f(m,n) \le \frac{1}{2}\max\{m,n\}$ for all m and n.

(c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

Soln. (Official solution): (a) For an arbitrary polygon P, let $S_b(P)$ and $S_w(P)$ denote the total area of the white part and the black part, respectively. Let A, B, C, D be the points $(0,0)$, $(0,m)$, $(n,0)$, (n,m), respectively. When m and n are of the same parity, the colouring of the rectangle $ABCD$ is centrally symmetric about its centre. Hence $S_w(ABC) = S_w(BCD)$ and $S_b(ABC) = S_b(BCD)$. Thus

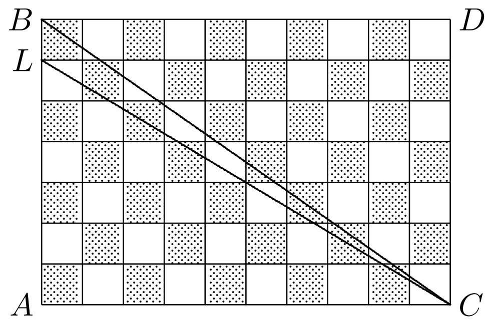
$$f(m,n) = |S_b(ABC) - S_w(ABC)| = \frac{1}{2}|S_b(ABCD) - S_w(ABCD)|.$$

Hence $f(m,n) = 0$ when both m and n are even and $f(m,n) = 1/2$ when both m and n are odd.



(b) If m and n are of the same parity the result follows from (a). Now suppose that m is odd and n is even. Let L be the point $(0, m-1)$. Then $f(m-1,n) = 0$, i.e. $S_b(ALC) = S_w(ALC)$. Thus

$$\begin{aligned} f(m,n) &= |S_b(ABC) - S_w(ABC)| = |S_b(LBC) - S_w(LBC)| \\ &\le \text{Area}\,(LBC) = \frac{n}{2} \le \frac{1}{2}\max\{m,n\}. \end{aligned}$$



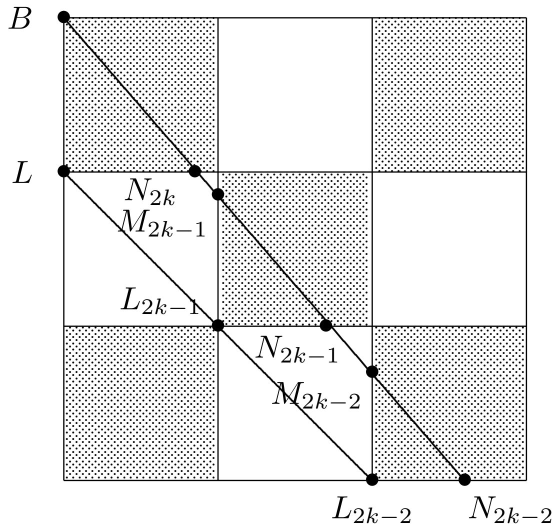
(c) As in (b), with m replaced by $2k+1$ and n by $2k$, we have

$$f(2k+1, 2k) = |S_b(LBC) - S_w(LBC)|.$$

The area of LBC is k. Without loss of generality suppose that the hypotenuse LC passes through white squares. Then the black part of LBC consists of several triangles BLN_{2k}, $M_{2k-1}L_{2k-1}N_{2k-1}$, …, $M_1L_1N_1$, each of them being similar to BAC. We have $LN_{2k} = 2k/(2k+1)$, $BL = 1$, $M_{2k-1}L_{2k-1} = (2k-1)/2k$, $M_{2k-2}L_{2k-2} = (2k-2)/2k$, and so on. Thus their total area is

$$S_b(LBC) = \frac{1}{2}\frac{2k}{2k+1}\left(\left(\frac{2k}{2k}\right)^2 + \left(\frac{2k-1}{2k}\right)^2 + \cdots + \left(\frac{1}{2k}\right)^2\right) = \frac{4k+1}{12}.$$

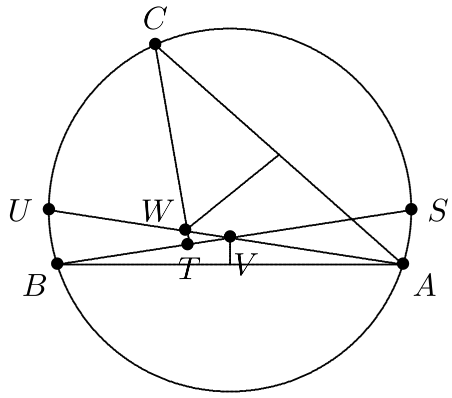
Hence $S_w(LBC) = k - \frac{4k+1}{12}$ and $f(2k+1, 2k) = (2k-1)/6$. Thus such a constant C cannot exist.



2. Angle A is the smallest in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that

$$AU = TB + TC.$$

Soln. (Official solution): (Note: The key is to notice that $AU = BS$ as in the solution.) Let the line BV meet the circle at the point S. Then $BS = AU$. Thus we only need to prove that $TC = TS$. Let $\angle ABS = x$ and $\angle VAC = y$. Then $\angle ACS = x$, $\angle VAB = x$ and $\angle WCA = y$. Thus $\angle BSC = \angle BAC = x + y$. Also $\angle TCS = x + y$. Thus $TC = TS$ as required.



3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions:

$$|x_1 + x_2 + \cdots + x_n| = 1$$

and

$$|x_i| \leq \frac{n+1}{2} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

Soln. (Note: This has something to do with rearrangement inequality. Try to go from the smallest to the largest by a series of steps, each of which is of length $\leq n + 1$.) Without loss of generality, let $x_1 + \cdots + x_n = 1$ and $x_1 \leq \cdots \leq x_n$. For each permutation P of the x_i's, let $S(P)$ be the corresponding sum. If I is the identity permutation and J is the permutation $x_n \ldots x_2 x_1$, then $S(I)$ is the largest and $S(J)$ is the smallest among all the sums. Let $r = (n + 1)/2$. Then $S(I) + S(J) = 2r$. If one of $S(I)$, $S(J)$ lies between $-r$ and r we are done. If not then $S(J) < -r$ and $S(I) > r$. First note that if we have a permutation p: $\ldots x_i x_j \ldots$ where $x_i > x_j$, and q is the permutation obtained by interchanging x_i and x_j, then $S(q) - S(p) \leq (n+1)$. We can go from J to I by a sequence of such operations, that is interchanging adjacent terms. Thus one of the intermediate permutations must have its sum lie between $-r$ and r.

4. An $n \times n$ matrix (square array) whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$, is called a *silver* matrix if, for each $i = 1, 2, \ldots, n$, the ith row and ith column together contain all elements of S. Show that

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

Soln. (Note: Part (a) is by a simple parity argument. The first construction is fairly standard. You should learn how to use it. The second construction is tricky and is adapted from the idea of Huah Cheng Jiann, Singapore's representative at the IMO.) (a) Let A be an $n \times n$ silver matrix. For each $i = 1, 2, \ldots, n$, let A_i be the set containing all the elements which are in the i^{th} row and the i^{th} column, excluding the diagonal element. Let x be an element which is not on the diagonal. (Such an element exists because there are only n entries on the diagonal but there are $2n-1$ elements.) If x is at the (i,j)-entry, then it is in A_i and A_j, which is called an x-pair. Thus x partitions the sets $A_1, \ldots, A_n$ into x-pairs and so n must be even. So there is no silver matrix of order 1997.

(b) *First construction:* **Suppose A is a $n \times n$ silver matrix.** Construct a $2n \times 2n$ silver matrix as follows. Put two copies of A on the diagonal. Then form an $n \times n$ Latin square B using the symbols $2n$ to $3n-1$ (each row and each column is a permutation of the symbols.) and another, say C, using the symbols $3n$ to $4n-1$. Use these as the off diagonal blocks. (See the matrix below)

$$\begin{pmatrix} A & B \\ C & A \end{pmatrix}$$

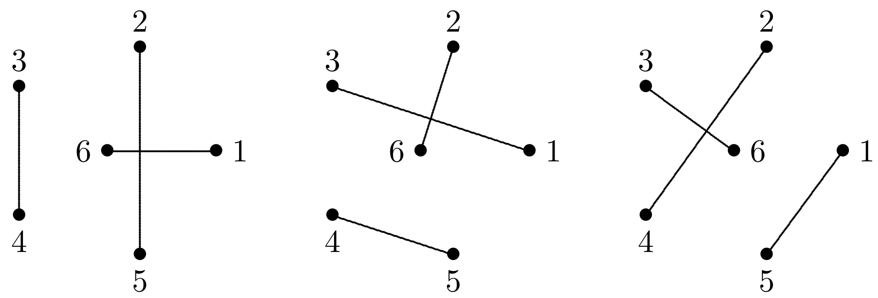
The matrix constructed is a $2n \times 2n$ silver matrix. Starting with the 2×2 silver matrix one can construct silver matrices of order 2^n for any natural number n. An $n \times n$ Latin square can be constructed by putting $1, 2, \ldots, n$ in the first row. Each subsequent row is obtained by taking the first element of the previous row and put it at the end. The matrices below are some examples.

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 2 & 4 & 5 \\ 3 & 1 & 5 & 4 \\ 6 & 7 & 1 & 2 \\ 7 & 6 & 3 & 1 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 2 & 4 & 5 & 8 & 9 & 10 & 11 \\ 3 & 1 & 5 & 4 & 9 & 10 & 11 & 8 \\ 6 & 7 & 1 & 2 & 10 & 11 & 8 & 9 \\ 7 & 6 & 3 & 1 & 11 & 8 & 9 & 10 \\ 12 & 13 & 14 & 15 & 1 & 2 & 4 & 5 \\ 13 & 14 & 15 & 12 & 3 & 1 & 5 & 4 \\ 14 & 15 & 12 & 13 & 6 & 7 & 1 & 2 \\ 15 & 12 & 13 & 14 & 7 & 6 & 3 & 1 \end{pmatrix}$$

Second construction: It can be shown that a silver matrix A of order $2n$ exist for all n. We need a few definitions. Define a 2-partition of the set of integers $N = \{1, 2, \ldots, 2n\}$ as a division of the set into pairwise disjoint 2-element subsets whose union is N. For example $\{\{1,2\},\{3,4\},\{5,6\}\}$ is a 2-partition of $\{1, 2, \ldots, 6\}$. Let a_{ij} denote the (i,j)-entry of A. Suppose k is an element which does not appear on the main diagonal of a $2n \times 2n$ matrix. Let $X_k = \{\{i,j\} : a_{ij} = k \text{ or } a_{ji} = k\}$. Then $k \in A_i$ and $k \in A_j$ if and only if $\{i,j\} \in X_k$, where A_i is as defined in part (a). Thus $k \in A_i$ for all i if and only if X_k is a 2-partition of $\{1, 2, \ldots, 2n\}$. If we have $(2n-1)$ 2-partitions of N, $B_1, \ldots, B_{2n-1}$ which are pairwise disjoint, then each pair of distinct integers $\{i,j\}$ with $1 \le i, j \le 2n$ is in exactly one of the

B_k's. A $2n \times 2n$ silver can be constructed as follows: Put $4n-1$ on the diagonal. For each $k = 1, 2, \ldots, 2n-1$, put $a_{ij} = k$ and $a_{ji} = k + 2n - 1$ if $i < j$ and $\{i, j\} \in B_k$. Then the resulting matrix is a silver matrix of order $2n$.

The desired 2-partitions can be constructed as follows. Consider a regular polygon with $2n - 1$ sides. Label the vertices as $1, 2, \ldots, 2n-1$ and the centre of the polygon as $2n$. Let B_k consists of the pair $\{k, 2n\}$ together with the pairs that are joined by lines which are perpendicular to the line joining k to $2n$. Then $B_1, \ldots, B_{2n-1}$ are the desired 2-partitions.



For example, when $n = 3$, we have the 2-partitions $B_1 = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$, $B_2 = \{\{2, 6\}, \{1, 3\}, \{4, 5\}\}$, $B_3 = \{\{3, 6\}, \{2, 4\}, \{1, 5\}\}$, $B_4 = \{\{4, 6\}, \{3, 5\}, \{1, 2\}\}$, $B_5 = \{\{5, 6\}, \{1, 4\}, \{2, 3\}\}$. The picture above shows B_1, B_2 and B_3. Put $a_{16} = a_{25} = a_{34} = 1$, $a_{61} = a_{52} = a_{43} = 6$, $a_{26} = a_{13} = a_{45} = 2$, $a_{62} = a_{31} = a_{54} = 7$, etc, we obtain the silver matrix

$$\begin{pmatrix} 11 & 4 & 2 & 5 & 3 & 1 \\ 9 & 11 & 5 & 3 & 1 & 2 \\ 7 & 10 & 11 & 1 & 4 & 3 \\ 10 & 8 & 6 & 11 & 2 & 4 \\ 8 & 6 & 9 & 7 & 11 & 5 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{pmatrix}.$$

5. Find all pairs (a, b) of integers, $a \geq 1$, $b \geq 1$ that satisfy the equation

$$a^{(b^2)} = b^a.$$

Soln. We need the following Lemma: If a, m, n are positive integers with m and n coprime, and $a^{m/n}$ is also an integer, then $a = k^n$ for some positive integer k.

Proof of the Lemma: Let $a^m = b^n$, where $b = a^{m/n}$. Then a and b have same prime factors. Let $a = p_1^{c_1} p_2^{c_2} ... p_s^{c_s}$ and $b = p_1^{d_1} p_2^{d_2} ... p_s^{d_s}$. It is not hard to see that n divides c_i for each i thus completing the proof.

For the solution of the problem, first note that $a = 1$ if and only if $b = 1$. So assume that both are not 1. Taking log, we have $b^2/a = \log b / \log a = t$. Thus $b^2 = at$ and $b = a^t$, whence $b^2 = a^{2t} = at$. Let $t = p/q$, where p, q are coprime. Since at is an integer, we have $q|a$. Moreover, if $q = 1$, then t is a positive integer and $a^{2t} = at$ cannot hold. Thus $q > 1$.

First consider the case q is odd. Since $a^{2p/q}$ is an integer and the pair $2p$ and q are coprime, $a^{1/q}$ is an integer by the Lemma. Also a is a multiple of q. Thus $a = \left(q^r k\right)^q$, where r and k are natural numbers and q, k coprime. Thus $a^{2t} = a^{2p/q} = q^{2rp}k^{2p}$ and $at = ap/q = q^{qr-1}pk^q$, whence $q^{2rp}k^{2p} = q^{qr-1}pk^q$. Since $q > 1$, we have $2rp = qr - 1$. This implies $2p < q$. If $k > 1$, then $2p \geq q$ which leads to a contradiction. Thus $k = 1$ and $p = 1$. This gives $1 = r(q-2)$ and $r = 1$, $q = 3$, whence $a = 27$, $b = 3$.

When q is even, $a^{2/q}$ is an integer. Thus $a = \left(q^r k\right)^{q/2}$, where r and k are natural numbers and q, k coprime. Thus $a^{2t} = a^{2p/q} = q^{rp}k^p$ and $at = ap/q = q^{(qr/2)-1}pk^{q/2}$, whence $q^{rp}k^p = q^{(qr/2)-1}pk^{q/2}$. Using the same argument as before, we have $p = 1$, $k = 1$, $q = 4$ and $r = 1$, and the corresponding answer is $a = 16$, $b = 2$.

(Note: The key step is that $a^{2t} = at \in \mathbb{Z}$, i.e., $a^{2p/q} = ap/q \in \mathbb{Z}$. Then we can conclude that q divides a and that a is a q power if q is odd and a can be written in the form given in the solution.)

6. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:

$$4; 2+2; 2+1+1; 1+1+1+1.$$

Prove that, for any integer $n \geq 3$:

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

Soln. (Official solution): Note that $2 = f(2) \leq f(n)$ for $n \geq 2$. Also, in an representation of $2n$ as a power of 2, the number 1's is always even. If there are $2k$ 1's, then the remaining terms, when divided by 2, give a representation of $n - k$. Thus

$$\begin{aligned} f(2n) &= 2 + (f(2) + \cdots + f(n)) \leq 2 + (n-1)f(n) \\ &\leq f(n) + (n-1)f(n) = nf(n), \quad \text{for } n = 2, 3, \ldots \end{aligned}$$

Consequently,

$$\begin{aligned} f(2^n) &\leq 2^{n-1}f(2^{n-1}) \leq 2^{n-1}2^{n-2}f(2^{n-2}) \\ &\leq \cdots \leq 2^{(n-1)+(n-2)+\cdots+1}f(2) = 2^{n(n-1)/2} \cdot 2. \end{aligned}$$

And since $2^{(n(n-1)/2} \cdot 2 < 2^{n^3/2}$ for $n \geq 3$, the upper estimate follows.

To find the lower estimate we use binary representations of numbers. Let $\alpha = (a_0, a_1, \ldots)$ be a sequence of integers with only a finite number of nonzero terms. Define $S(\alpha) = a_0 2^0 + a_1 2^1 + \cdots$. For two sequences $\alpha = (a_0, a_1, \ldots)$ and $\beta = (b_0, b_1, \ldots)$, define $\alpha + \beta = (a_0 + b_0, a_1 + b_1, \ldots)$. Then $S(\alpha + \beta) = S(\alpha) + S(\beta)$. For any integer n

let $n = a_0 2^0 + a_1 2^1 + \cdots + a_k 2^k$ be the binary representation of n, i.e., $a_i = 0$, or 1 for $i = 1, 2, \ldots, k$. Let $\text{bin}(n) = (a_0, a_1, \ldots, a_k, 0, 0, \ldots)$. Then $S(\text{bin}(n)) = n$.

Now we are ready to find the lower estimate. We start by proving that

$$f(2^{n+3}) \geq 2^{2n-1} f(2^n). \tag{$*$}$$

Let $\alpha = (a_0, a_1, \ldots)$ be a representation of 2^n. We will construct 2^{2n-1} different representations of 2^{n+3}. First, the sequence $\beta(\alpha) = (0, 0, a_0, a_1, \ldots)$ is a representation of 2^{n+2}. Let $\gamma(x) = (0, x, 0, 0, \ldots) + \text{bin}(2^{n+1} - 2x)$ and $\delta(y) = (y, 0, 0, \ldots) + \text{bin}(2^{n+1} - y)$, where $0 \leq x \leq 2^n$ and $0 \leq y \leq 2^{n+1}$ are both even. Note the first entry of $\delta(y)$ is always y and the first two entries of $\gamma(x)$ are always 0 and x. We have $S(\gamma(x)) = S(\delta(x)) = 2^{n+1}$. Define $F(\alpha, x, y) = \beta(\alpha) + \gamma(x) + \delta(y)$. Then $F(\alpha, x, y)$ is a representation of 2^{n+3}. We need to show that $F(\alpha, x, y) \neq F(\alpha', x', y')$ if $(\alpha, x, y) \neq (\alpha', x'y')$. To this end, we let $F(\alpha, x, y) = F(\alpha', x'y')$. Then comparing the first entries, we have $y = y'$. Thus $\beta(\alpha) + \gamma(x) = \beta(\alpha') + \gamma(x')$. The second entry of the left hand side is always x and that of the right hand side is always x'. Thus $x = x'$. This then implies $\alpha = \alpha'$. Thus the proof is complete. The inequality $(*)$ then follows readily.

We now complete the proof by induction. First we assume that for some $n > 6$ we have $f(2^n) > 2^{n^2/4}$. Then

$$f(2^{n+3}) \geq 2^{2n-1} f(2^n) > 2^{2n-1} 2^{n^2/4} \geq 2^{(n+3)^2/4}.$$

Thus the inequality holds for $n + 3$ as well. To complete the proof we need to check the cases for $n = 3, 4, \ldots, 9$. This can be done easily using the fact that f is strictly increasing and the details are left to the readers.