40th International Mathematical Olympiad

Bucharest, Romania, July 1999.

1. Determine all finite sets S of at least three points in the plane which satisfy the following condition:

> for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

Soln. Let G be the centre of gravity of the set S. Since the perpendicular bisector of the line joining every pair of points is an axis of symmetry, G lies on the perpendicular bisector. Thus the perpendicular bisectors meet at a common point G. Thus the points lie on a circle. Let a, b, c be three consecutive points. Since the perpendicular of ac is an axis of symmetry, b must lie on it. Thus the lengths of ab and bc are equal. Thus the points are the vertices of a regular polygon.

Second soln. One can show easily that the boundary of the convex hull of the points form a regular polygon as in the second half of the previous proof. The only thing left to do is to prove that there is no point in the interior. This can be proved by contradiction.

2. Let n be a fixed integer, with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1\leq i<j\leq n} x_i x_j (x_i^2 + x_j^2) \leq C\Big(\sum_{1\leq i\leq n} x_i\Big)^4$$

holds for all real numbers $x_1, \ldots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

Soln. The inequality is symmetric and homogeneous, so we can assume $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ and $\sum x_i = 1$. In this case we have to maximize the sum

$$F(x_1, \ldots, x_n) = \sum_{i<j} x_i x_j (x_i^2 + x_j^2).$$

Let x_{k+1} be the last nonzero coordinate and we assume that $k \geq 2$. We shall replace $x = (x_1, \ldots, x_k, x_{k+1}, 0, \ldots, 0)$ with

$$x' = (x_1, \ldots, x_{k-1}, x_k + x_{k+1}, 0, \ldots, 0)$$

to increase the value of F as shown below:

$$\begin{aligned}F(x')-F(x) &= x_kx_{k+1}[3(x_k+x_{k+1})\sum_{i=1}^{k-1}x_i - x_k^2 - x_{k+1}^2]\\ &= x_kx_{k+1}[3(x_k+x_{k+1})(1-x_k-x_{k+1}) - x_k^2 - x_{k+1}^2]\\ &= x_kx_{k+1}(x_k+x_{k+1})(3-4(x_k+x_{k+1})) + 2x_kx_{k+1}].\end{aligned}$$

From

$$1 \geq x_1 + x_k + x_{k+1} \geq \frac{1}{2}(x_k+x_{k+1}) + x_k + x_{k+1}$$

it follows that $2/3 \geq x_k + x_{k+1}$, and therefore $F(x') - F(x) > 0$. After several such substitutions, we have

$$\begin{aligned}F(x) &\leq F(a,b,0,\dots,0) = ab(a^2+b^2)\\ &= \frac{1}{2}(2ab)(1-2ab) \leq \frac{1}{8} = F(1/2,1/2,0,\dots,0).\end{aligned}$$

Thus $C = 1/8$ and equality occurs if and only if two of the x_i's are equal (possibly zero) and the remaining variables are zero.

3. Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are *adjacent* if they have a common side. N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square. Determine the smallest possible value of N.

Soln. Let $n = 2k$. First colour the board black and white like a chessboard. We say that a set S of cells is *covered* by a set M of marked cells if every cell in S is neighbour to a cell in M. Let $f_w(n)$ be the minimal number of white cells that must be marked so as to cover all the black cells. Define similarly $f_b(n)$. Due to symmetry of the chessboard (n is even), we have

$$f_w(n) = f_b(n).$$

Since black cells can only be covered by marked white cells and vice versa, we have

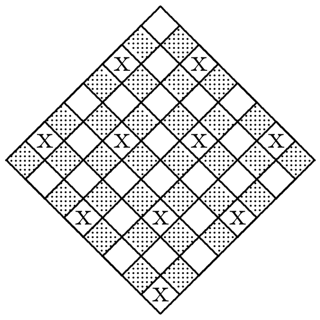
$$N = f_w(n) + f_b(n).$$

Place the board so that the longest black diagonal is horizontal. Now consider the horizontal rows of white cells. Mark the odd cells of the first row, third row, fifth row, etc. (The figure illustrates the case $n = 8$). Call this set of cells M. The number of cells in M is

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

It is easy to see that the black cells are covered by M. Thus

$$f_w(n) \leq \frac{k(k+1)}{2}$$



Now consider the cells in M. No two of them have a common black neighbour. So we need to mark at least $k(k+1)/2$ black cells in order to cover M. Therefore

$$f_b(n) \geq \frac{k(k+1)}{2}.$$

Hence

$$f_b(n) = f_w(n) = k(k+1)/2.$$

Thus

$$N = f_b(n) + f_w(n) = k(k+1).$$

4. Determine all pairs (p, q) of positive integers such that p is prime, $n \leq 2p$, and $(p-1)^n + 1$ is divisible by n^{p-1}.

Soln. Clearly $(1, p)$ and $(2, 2)$ are solutions and for other solutions we have $p \geq 3$. Now assume that $n \geq 2$ and $p \geq 3$. Since $(p-1)^n + 1$ is odd and is divisible by n^{p-1}, n must be odd. Thus $n < 2p$. Let q be the smallest prime divisor of n. From $q \mid (p-1)^n + 1$, we have

$$(p-1)^n \equiv -1 \pmod q \quad \text{and} \quad \gcd(q, p-1) = 1.$$

But $\gcd(n, q-1) = 1$ (from the choice of q), there exist integers u and v such that $un + v(q-1) = 1$, whence

$$p - 1 \equiv (p-1)^{un}(p-1)^{v(q-1)} \equiv (-1)^u 1^v \equiv -1 \pmod q,$$

because u must be odd. This shows $q \mid p$ and therefore $q = p$. Hence $n = p$. Now

$$\begin{aligned} p^{p-1} &\mid (p-1)^p + 1 \\ &= p^2\left(p^{p-2} - \binom{p}{1}p^{p-3} + \cdots + \binom{p}{p-3}p - \binom{p}{p-2} + 1\right) \end{aligned}$$

Since every term in the bracket except the last is divisible by p, we have $p - 1 \leq 2$. Thus $p = 3 = n$. Indeed $(3, 3)$ is a solution.

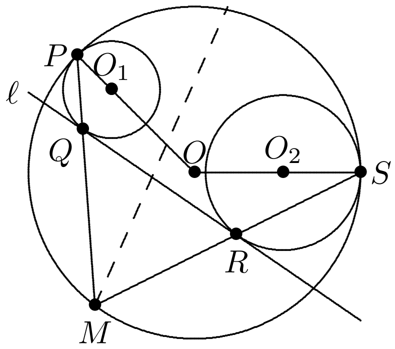
In conclusion, the only solutions are $(1, p), (2, 2), (3, 3)$.

5. Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the centre of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B. The lines MA and MB meet Γ_1 at C and D, respectively.

Prove that CD is tangent to Γ_2.

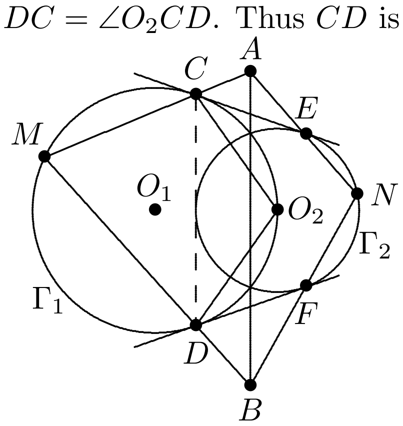
Soln. We first prove a lemma.

Lemma: Let Γ_1 and Γ_2 be two circles such that one does not contain the other. Let Γ be a circle containing both Γ_1, Γ_2 and touches Γ_1 at P, Γ_2 at S. Let ℓ be a chord of Γ which is tangent to Γ_1 at Q and Γ_2 at R, with both circles on the same side of the chord. Then PQ and SR meet at a point M which is on Γ. Furthermore M is on the radical axis of Γ_1 and Γ_2.



Proof: Let O, O_1, O_2 be the centres of $\Gamma, \Gamma_1, \Gamma_2$, respectively. Produce PQ to meet Γ at X. Then it is easy to see that $O_1Q||OX$. Thus the tangent at X is parallel to ℓ. Produce SR to meet Γ at Y. A similar consideration shows that the tangent at Y is also parallel to ℓ. Thus $X = Y = M$. Since ℓ is parallel to the tangent at M, we have $\angle SPQ = 180^\circ - \angle QRS$. Thus $PQRS$ is cyclic. Hence the powers of M with respect to Γ_1 and Γ_2 are equal and M is on the radical axis of both. This completes the proof of the lemma.

Let E be the intersection of NA with Γ_2 and F be the intersection of NB with Γ_2. From the lemma, and since the radical axis in this case is the common chord AB, we known that CE and DF are both common tangents of Γ_1 and Γ_2. Thus O_1O_2 is the perpendicular bisector of CD, i.e., $O_2C = O_2D$ and $\angle O_2CD = \angle O_2DC$. Since CE is tangent to Γ_1, $\angle ECO_2 = \angle O_2DC = \angle O_2CD$. Thus CD is tangent to Γ_2.



6. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

Soln. Let $A = \mathrm{Im} f$ and $c = f(0)$. By putting $x = y = 0$, we get $f(-c) = f(c) + c - 1$, so $c \neq 0$.

It is easy to find the restriction f to A. Take $x = f(y)$ to obtain

$$f(x) = \frac{c+1}{2} - \frac{x^2}{2} \quad \text{for all} \quad x \in A. \tag{1}$$

The main step is to show that $A - A = \mathbb{R}$. Indeed, for $y = 0$, we get:

$$\{f(x-c) - f(x) \mid x \in \mathbb{R}\} = \{cx + f(c) - 1 \mid x \in \mathbb{R}\} = \mathbb{R}$$

because $c \neq 0$.

With this we conclude that the given functional equation is equivalent to the following:

$$f(x-y) = f(y) + xy + f(x) - 1, \quad \text{for all } x, y \in \mathbb{R}.$$

By putting $y = 0$, we get $f(0) = 1$. Replacing y by x, we have

$$1 = 2f(x) + x^2 - 1, \quad \text{i.e., } f(x) = 1 - \frac{x^2}{2}.$$

It is easy to check that this function satisfies the given functional equation.