Singapore International Mathematical Olympiad

National Team Selection Tests 1994/95 - 2003/2004

1994/95

1.1.* Let $\mathbb{N} = \{1, 2, 3, ...\}$ be the set of all natural numbers and $f : \mathbb{N} \to \mathbb{N}$ be a function. Suppose $f(1) = 1$, $f(2n) = f(n)$ and $f(2n+1) = f(2n) + 1$ for all natural numbers n.

(i) Calculate the maximum value M of $f(n)$ for $n \in \mathbb{N}$ with $1 \leq n \leq 1994$.

(ii) Find all $n \in \mathbb{N}$, with $1 \leq n \leq 1994$, such that $f(n) = M$.

1.2. ABC is a triangle with $\angle A > 90^o$. On the side BC, two distinct points P and Q are chosen such that $\angle BAP = \angle PAQ$ and $BP \cdot CQ = BC \cdot PQ$. Calculate the size of $\angle PAC$.

1.3. In a dance, a group S of 1994 students stand in a big circle. Each student claps the hands of each of his two neighbours a number of times. For each student x, let $f(x)$ be the total number of times x claps the hands of his neighbours. As an example, suppose there are 3 students A, B and C. A claps hand with B two times, B claps hand with C three times and C claps hand with A five times. Then $f(A) = 7, f(B) = 5$ and $f(C) = 8$.

(i) Prove that $\{f(x) \mid x \in S\} \neq \{n \mid n \text{ is an integer}, 2 \leq n \leq 1995\}$.

(ii) Find an example in which

$$\{f(x) \mid x \in S\} = \{n \mid n \text{ is an integer}, n \neq 3, 2 \leq n \leq 1996\}.$$

2.1. Let $f(x) = \frac{1}{1+x}$ where x is a positive real number, and for any positive integer n, let

$$g_n(x) = x + f(x) + f(f(x)) + \cdots + f(f(\ldots f(x))),$$

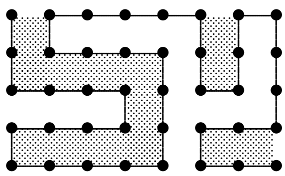
the last term being f composed with itself n times. Prove that

(i) $g_n(x) > g_n(y)$ if $x > y > 0$.

(ii) $g_n(1) = \frac{F_1}{F_2} + \frac{F_2}{F_3} + \cdots + \frac{F_{n+1}}{F_{n+2}}$, where $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

2.2. Let ABC be an acute-angled triangle. Suppose that the altitude of $\triangle ABC$ at B intersects the circle with diameter AC at P and Q, and the altitude at C intersects the circle with diameter AB at M and N. Prove that P, Q, M and N lie on a circle.

2.3. Show that a path on a rectangular grid which starts at the northwest corner, goes through each point on the grid exactly once, and ends at the southeast corner divides the grid into two equal halves: (a) those regions opening north or east; and (b) those regions opening south or west.



(*The figure above shows a path meeting the conditions of the problem on a* 5×8 *grid. The shaded regions are those opening north or east while the rest open south or west.*)

1995/96

1.1. Let P be a point on the side AB of a square $ABCD$ and Q a point on the side BC. Let H be the foot of the perpendicular from B to PC. Suppose that $BP = BQ$. Prove that QH is perpendicular to HD.

1.2. For each positive integer k, prove that there is a perfect square of the form $n2^k - 7$, where n is a positive integer.

1.3. Let $S = \{0, 1, 2, \ldots, 1994\}$. Let a and b be two positive numbers in S which are relatively prime. Prove that the elements of S can be arranged into a sequence $s_1, s_2, s_3, \ldots, s_{1995}$ such that $s_{i+1} - s_i \equiv \pm a \text{ or } \pm b \pmod{1995}$ for $i = 1, 2, \ldots, 1994$.

2.1. Let C, B, E be three points on a straight line l in that order. Suppose that A and D are two points on the same side of l such that

(i) $\angle ACE = \angle CDE = 90^o$ and

(ii) CA = CB = CD.

Let F be the point of intersection of the segment AB and the circumcircle of $\triangle ADC$. Prove that F is the incentre of $\triangle CDE$.

2.2. Prove that there is a function f from the set of all natural numbers to itself such that for any natural number n, $f(f(n)) = n^2$.

2.3. Let S be a sequence $n_1, n_2, \ldots, n_{1995}$ of positive integers such that $n_1 + \cdots + n_{1995} = m < 3990$. Prove that for each integer q with $1 \leq q \leq m$, there is a sequence $n_{i_1}, n_{i_2}, \ldots, n_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq 1995$, $n_{i_1} + \cdots + n_{i_k} = q$ and k depends on q.

1996/97

1.1. Let ABC be a triangle and let D, E and F be the midpoints of the sides AB, BC and CA respectively. Suppose that the angle bisector of $\angle BDC$ meets BC at the point M and the angle bisector of $\angle ADC$ meets AC at the point N. Let MN and CD intersect at O and let the line EO meet AC at P and the line FO meet BC at Q. Prove that $CD = PQ$.

1.2. Let a_n be the number of n-digit integers formed by 1, 2 and 3 which do not contain any consecutive 1's. Prove that a_n is equal to $(\frac{1}{2} + \frac{1}{\sqrt{3}})(\sqrt{3} + 1)^n$ rounded off to the nearest integer.

1.3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function from the set $\mathbb{R}$ of real numbers to itself. Find all such functions f satisfying the two properties:

(a) $f(x + f(y)) = y + f(x)$ for all $x, y \in \mathbb{R}$,

(b) the set $\left\{ \dfrac{f(x)}{x} \ : \ x \text{ is a nonzero real number} \right\}$ is finite.

2.1. Four integers a_0, b_0, c_0, d_0 are written on a circle in the clockwise direction. In the first step, we replace a_0, b_0, c_0, d_0 by a_1, b_1, c_1, d_1, where $a_1 = a_0 - b_0, b_1 = b_0 - c_0, c_1 = c_0 - d_0, d_1 = d_0 - a_0$. In the second step, we replace a_1, b_1, c_1, d_1 by a_2, b_2, c_2, d_2, where $a_2 = a_1 - b_1, b_2 = b_1 - c_1, c_2 = c_1 - d_1, d_2 = d_1 - a_1$. In general, at the kth step, we have numbers a_k, b_k, c_k, d_k on the circle where $a_k = a_{k-1} - b_{k-1}, b_k = b_{k-1} - c_{k-1}, c_k = c_{k-1} - d_{k-1}, d_k = d_{k-1} - a_{k-1}$. After 1997 such replacements, we set $a = a_{1997}, b = b_{1997}, c = c_{1997}, d = d_{1997}$. Is it possible that all the numbers $|bc - ad|, |ac - bd|, |ab - cd|$ are primes ? Justify your answer.

2.2. For any positive integer n, evaluate

$$\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i+1}{i},$$

where $\binom{m}{k} = \dfrac{m!}{k!(m-k)!}$ and $\lfloor \frac{n+1}{2} \rfloor$ is the greatest integer less than or equal to $\frac{n+1}{2}$.

2.3. Suppose the numbers $a_0, a_1, a_2, \ldots, a_n$ satisfy the following conditions:

$$a_0 = \frac{1}{2}, \quad a_{k+1} = a_k + \frac{1}{n} a_k^2 \quad \text{for} \quad k = 0, 1, \ldots, n-1.$$

Prove that $1 - \dfrac{1}{n} < a_n < 1$.

1997/98

1.1. Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$ and $EF = FA$. Prove that

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

When does the equality occur?

1.2. Let $n \geq 2$ be an integer. Let S be a set of n elements and let A_i, $1 \leq i \leq m$, be distinct subsets of S of size at least 2 such that

$$A_i \cap A_j \neq \emptyset, \; A_i \cap A_k \neq \emptyset, \; A_j \cap A_k \neq \emptyset \quad \text{imply} \quad A_i \cap A_j \cap A_k \neq \emptyset.$$

Show that $m \leq 2^{n-1} - 1$.

1.3. Suppose $f(x)$ is a polynomial with integer coefficients satisfying the condition

$$0 \leq f(c) \leq 1997 \quad \text{for each } c \in \{0, 1, \ldots, 1998\}.$$

Is is true that $f(0) = f(1) = \cdots = f(1998)$?

2.1. Let I be the centre of the inscribed circle of the non-isosceles triangle ABC, and let the circle touch the sides BC, CA, AB at the points A_1, B_1, C_1 respectively. Prove that the centres of the circumcircles of $\triangle AIA_1$, $\triangle BIB_1$ and $\triangle CIC_1$ are collinear.

2.2. Let $a_1 \geq \cdots \geq a_n \geq a_{n+1} = 0$ be a sequence of real numbers. Prove that

$$\sqrt{\sum_{k=1}^{n} a_k} \leq \sum_{k=1}^{n} \sqrt{k}\left(\sqrt{a_k} - \sqrt{a_{k+1}}\right).$$

2.3. Let p and q be distinct positive integers. Suppose p^2 and q^3 are terms of an infinite arithmetic progression whose terms are positive integers. Show that the arithmetic progression contains the sixth power of some integer.

1998/99

1.1. Find all integers m for which the equation

$$x^3 - mx^2 + mx - (m^2 + 1) = 0$$

has an integer solution.

1.2. Is it possible to use 2×1 dominoes to cover a $2k \times 2k$ checkerboard which has 2 squares, one of each colour, removed?

1.3. Find the number of 16-tuples $(x_1, x_2, \ldots, x_{16})$ such that

(i) $x_i = \pm 1$ for $i = 1, \ldots, 16$,
(ii) $0 \leq x_1 + x_2 + \cdots + x_r < 4$, for $r = 1, 2, \ldots, 15$,
(iii) $x_1 + x_2 + \cdots + x_{10} = 4$.

2.1. Let M and N be two points on the side BC of a triangle ABC such that $BM = MN = NC$. A line parallel to AC meets the segments AB, AM and AN at the points D, E and F respectively. Prove that $EF = 3DE$.

2.2. Find all possible values of

$$\left\lfloor \frac{x-p}{p} \right\rfloor + \left\lfloor \frac{-x-1}{p} \right\rfloor,$$

where x is a real number and p is a nonzero integer.

Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.

2.3. Let $f(x) = x^{1998} - x^{199} + x^{19} + 1$. Prove that there is an infinite set of prime numbers, each dividing at least one of the integers $f(1), f(2), f(3), f(4), \cdots$.

1999/2000

1.1. In a triangle ABC, $AB > AC$, the external bisector of angle A meets the circumcircle of triangle ABC at E, and F is the foot of the perpendicular from E onto AB. Prove that $2AF = AB - AC$.

1.2. Find all prime numbers p such that $5^p + 12^p$ is a perfect square.

1.3. There are n blue points and n red points on a straight line. Prove that the sum of all distances between pairs of points of the same colour is less than or equal to the sum of all distances between pairs of points of different colours.

2.1. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such for any $x, y \in \mathbb{R}$,

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2).$$

2.2. In a triangle ABC, $\angle C = 60°$, D, E, F are points on the sides BC, AB, AC respectively, and M is the intersection point of AD and BF. Suppose that $CDEF$ is a rhombus. Prove that $DF^2 = DM \cdot DA$.

2.3. Let n be any integer ≥ 2. Prove that $\sum 1/pq = 1/2$, where the summation is over all integers p, q which satisfy $0 < p < q \leq n$, $p + q > n$, $(p, q) = 1$.

2000/2001

1.1. Let a, b, c, d be four positive integers such that each of them is a difference of two squares of positive integers. Prove that $abcd$ is also a difference of two squares of positive integers.

1.2. Let P, Q be points taken on the side BC of a triangle ABC, in the order B, P, Q, C. Let the circumcircles of $\triangle PAB, \triangle QAC$ intersect at $M(\neq A)$ and those of $\triangle PAC$, $\triangle QAB$ at N. Prove that A, M, N are collinear if and only if P and Q are symmetric in the midpoint A' of BC.

1.3. A game of Jai Alai has eight players and starts with players P_1 and P_2 on court and the other players $P_3, P_4, P_5, P_6, P_7, P_8$ waiting in a queue. After each point is played, the loser goes to the end of the queue; the winner adds 1 point to his score and stays on the court; and the player at the head of the queue comes on to contest the next point. Play continues until someone has scored 7 points. At that moment, we observe that a total of 37 points have been scored by all eight players. Determine who has won and justify your answer.

2.1. In the acute triangle ABC, let D be the foot of the perpendicular from A to BC, let E be the foot of the perpendicular from D to AC, and let F be a point on the line segment DE. Prove that AF is perpendicular to BE if and only if $FE/FD = BD/DC$.

2.2. Determine all the integers $n > 1$ such that

$$\sum_{i=1}^{n} x_i^2 \geq x_n \sum_{i=1}^{n-1} x_i$$

for all real numbers $x_1, x_2, \ldots, x_n$.

2.3. Let $L(n)$ denote the least common multiple of $\{1, 2 \ldots, n\}$.

(i) Prove that there exists a positive integer k such that

$$L(k) = L(k+1) = \cdots = L(k+2000).$$

(ii) Find all m such that $L(m+i) \neq L(m+i+1)$ for all $i = 0, 1, 2$.

2001/2002

1.1. Let A, B, C, D, E be five distinct points on a circle Γ in the clockwise order and let the extensions of CD and AE meet at a point Y outside Γ. Suppose X is a point on the extension of AC such that XB is tangent to Γ at B. Prove that $XY = XB$ if and only if XY is parallel DE.

1.2. Let n be a positive integer and $(x_1, x_2, \ldots, x_{2n})$, $x_i = 0$ or 1, $i = 1, 2, \ldots, 2n$ be a sequence of $2n$ integers. Let S_n be the sum

$$S_n = x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n}.$$

If O_n is the number of sequences such that S_n is odd and E_n is the number of sequences such that S_n is even, prove that

$$\frac{O_n}{E_n} = \frac{2^n - 1}{2^n + 1}.$$

1.3. For every positive integer n, show that there is a positive integer k such that

$$2k^2 + 2001k + 3 \equiv 0 \pmod{2^n}.$$

2.1. Let x_1, x_2, x_3 be positive real numbers. Prove that

$$\frac{(x_1^2 + x_2^2 + x_3^2)^3}{(x_1^3 + x_2^3 + x_3^3)^2} \leq 3.$$

2.2. For each real number x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x. For example $\lfloor 2.8 \rfloor = 2$. Let $r \geq 0$ be a real number such that for all integers m, n, $m|n$ implies $\lfloor mr \rfloor | \lfloor nr \rfloor$. Prove that r is an integer.

2.3. Find all functions $f : [0, \infty) \longrightarrow [0, \infty)$ such that $f(f(x)) + f(x) = 12x$, for all $x \geq 0$.

2002/2003

1.1. Determine whether there exists a positive integer n such that the sum of the digits of n^2 is 2002.

1.2. Three chords AB, CD and EF of a circle intersect at the midpoint M of AB. Show that if CE produced and DF produced meet the line AB at the points P and Q respectively, then M is also the midpoint of PQ.

1.3. In how many ways can n^2 distinct real numbers be arranged into an $n \times n$ array (a_{ij}) such that $\max_j \min_i a_{ij} = \min_i \max_j a_{ij}$?

2.1. Let $A = \{3 + 10k,\ 6 + 26k,\ 5 + 29k,\ k = 1, 2, 3, 4, \cdots\}$. Determine the smallest positive integer r such that there exists an integer b with the property that the set $B = \{b + rk,\ k = 1, 2, 3, 4, \cdots\}$ is disjoint from A.

2.2. Let M be a point on the diameter AB of a semicircle Γ. The perpendicular at M meets the semicircle Γ at P. A circle inside Γ touches Γ and is tangent to PM at Q and AM at R. Prove that $PB = RB$.

2.3. Determine all functions $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, such that

$$f(m + f(f(n))) = -f(f(m + 1)) - n$$

for all integers m and n.

2003/2004

1.1. Let N be the fourth root of a product of 8 consecutive positive integers. Prove that the greatest integer less than or equal to N is even.

1.2. Let Γ be a circle with center I, the incenter of triangle ABC. Let D, E, F be points of intersection of Γ with the lines from I that are perpendicular to the sides BC, CA, AB respectively. Prove that AD, BE, CF are concurrent.

1.3. Find all pairs of integers (x, y) satisfying $x^5 + y^5 = (x + y)^3$.

2.1. Let A, B, C, D be four distinct points arranged in order on a circle. The tangent to the circle at A meets the ray CB at K and the tangent to the circle at B meets the ray DA at H. Suppose $BK = BC$ and $AH = AD$. Prove that the quadrilateral $ABCD$ is a trapezium.

2.2. Determine the smallest constant $k > 0$ such that

$$\frac{ab}{a + b + 2c} + \frac{bc}{b + c + 2a} + \frac{ca}{c + a + 2b} \leq k(a + b + c),$$

for all $a, b, c > 0$.

2.3. Consider an $n \times n$ square lattice with points colored either black or white. A *square path* is a closed path in the shape of a square with edges parallel to the edges of the lattice. Let $M(n)$ be the minimum number of black points needed for an $n \times n$ square lattice so that every square path has at least one black point on it. Prove that

$$\frac{2}{7}(n - 1)^2 \leq M(n) \leq \frac{2}{7}n^2.$$

(**The numbering 1.1 refers to the first question of the selection test in the first day, while 2.1 refers to the first question of the selection test in the second day.*)

Solutions to National Team Selection Tests

Prepared by Tay Tiong Seng and Wong Yan Loi

1994/95

1.1 It can be proved by induction that $f(n)$ is the number of ones in the binary representation of n.

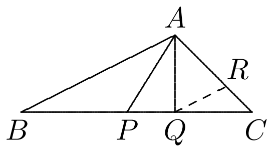
(i) There can be at most 10 ones in the binary representation of a natural number if it is less than or equal to $1994 = 11111001010_{(2)}$. Hence $M = 10$.

(ii) For any natural number n less than or equal to 1994, $f(n) = 10$ if and only if n is

$$\begin{aligned} 1023 &= 1111111111_{(2)}, \\ 1535 &= 10111111111_{(2)}, \\ 1791 &= 11011111111_{(2)}, \\ 1919 &= 11101111111_{(2)}, \\ 1983 &= 11110111111_{(2)}. \end{aligned}$$

1.2. **Stewart's theorem.** In $\triangle ABC$, D is a point on BC such that AD bisects $\angle A$. Then $AB : BD = AC : CD$.

1st solution



Applying Stewart's theorem to $\triangle ABQ$, we have $\dfrac{AB}{AQ} = \dfrac{BP}{PQ}$.

Given $BP \cdot CQ = BC \cdot PQ$, it follows that $\dfrac{BC}{CQ} = \dfrac{AB}{AQ}$.

Now let R be the point on AC such that QR is parallel to BA.

Then $\dfrac{AB}{RQ} = \dfrac{BC}{CQ} = \dfrac{AB}{AQ}$.

Hence $RQ = AQ$ and $\angle QAR = \angle QRA$.

Therefore $\angle PAC = \angle PAQ + \angle QAR = \frac{1}{2}(\angle BAQ + \angle QAR + \angle QRA) = \frac{\pi}{2}$.

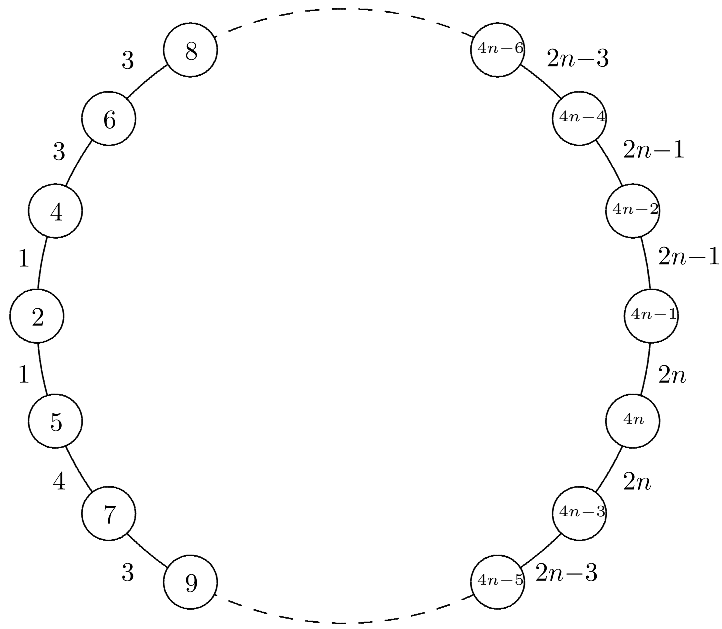
2nd solution

Since $\dfrac{CB}{CQ} = \dfrac{PB}{PQ} = \dfrac{AB}{AQ}$, by Stewart's theorem, AC is the external angle bisector of $\angle BAQ$.

Hence $\angle PAC = \dfrac{\pi}{2}$.

1.3. (i) Note that twice the total number of clappings is equal to $\sum_{x \in S} f(x)$ which cannot be the odd number $2 + 3 + 4 + \cdots + 1995$.

(ii) Let $n \geq 2$. For a group S_n of $4n - 2$ students, the following configuration gives an example in which $\{f(x) \mid x \in S_n\} = \{2, 4, 5, ..., 4n\}$.



Each circle in the diagram represents a student x and the number in the circle represents $f(x)$. The number on each edge represents the number of times the two adjacent students clap hands with each other. Taking $n = 499$ gives an example of the problem.

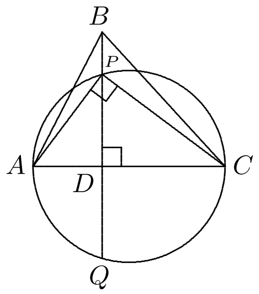
2.1. (i) Denote the function $f(x)$ composed with itself n times by $f^{(n)}(x)$. Also let $g_0(x)$ be the identity function. Note that $f^{(2)}(x)$ is strictly increasing for $x > 0$. We shall prove by induction on n that $g_n(x)$ is strictly increasing for $x > 0$. It can easily be checked that $g_1(x)$ is strictly increasing for $x > 0$.
Suppose for $n \geq 2$, $g_1(x), ..., g_{n-1}(x)$ are strictly increasing. Let $x > y > 0$. We have

$$\begin{aligned} g_n(x) - g_n(y) &= (x-y) + (f(x)-f(y)) + (f^{(2)}(x) - f^{(2)}(y)) + \cdots + (f^{(n)}(x) - f^{(n)}(y)) \\ &= (g_1(x) - g_1(y)) + (g_{n-2}(f^{(2)}(x)) - g_{n-2}(f^{(2)}(y))) \ > \ 0. \end{aligned}$$

By induction, $g_n(x)$ is strictly increasing.

(ii) Note that $\dfrac{F_1}{F_2} = 1$ and $f(\dfrac{F_i}{F_{i+1}}) = \dfrac{F_{i+1}}{F_{i+2}}$. Hence $\dfrac{F_1}{F_2} + \cdots + \dfrac{F_{n+1}}{F_{n+2}} = g_n(1)$.

2.2. Since $\triangle ADP$ is similar to $\triangle APC$, we have $AP/AD = AC/AP$. Hence $AP^2 = AD \cdot AC = (BD \cot A) \cdot AC = 2(ABC) \cot A$, where (ABC) is the area of $\triangle ABC$. Similarly, $AM^2 = 2(ABC) \cot A$. Hence $AP = AQ = AM = AN = \sqrt{2(ABC) \cot A}$. This shows that P, Q, M, N lie on the circle centered at A with radius $\sqrt{2(ABC) \cot A}$.



2.3. Let such a path be given. First the following facts are observed.
(i) The number of edges of the path is $nm - 1$.

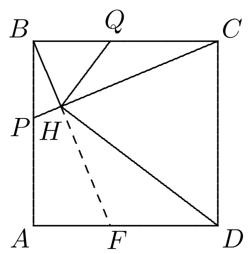
(ii) By induction, each region with s squares is adjacent to $2s+1$ edges of the path.
(iii) Each edge on the north or east side of the grid which is not included in the path corresponds to exactly one shaded region.
Let the number of shaded regions be k and let $s_1, s_2, ..., s_k$ be the number of squares in each of these regions. From (iii), it follows that the number of edges of the path on the north and east side of the grid is $(m-1)+(n-1)-k$. Hence by (ii), the total number of edges of the path is $\sum_{i=1}^{k}(2s_i+1) \ + \ [(m-1)+(n-1)-k]$. By (i), we have

$$\sum_{i=1}^{k}(2s_i+1) \ + \ [(m-1)+(n-1)-k] = nm-1.$$

From this the total number of shaded squares is $\displaystyle\sum_{i=1}^{k} s_i = \frac{1}{2}(m-1)(n-1)$.

This problem appears in the American Mathematical Monthly. (See The American Mathematical Monthly, Vol.104, No.6, June-July 1997, p572-573.)

1995/96



1.1. Let BH intersect AD at F. Then $\triangle AFB$ is congruent to $\triangle BPC$. Hence $AF = BP = BQ$. Therefore $FD = QC$ and $QCDF$ is a rectangle. Since $\angle CHF = 90^o$, the circumcircle of the rectangle $QCDF$ passes through H. As QD is also a diameter of this circle, we have $\angle QHD = 90^o$.

1.2. Suppose there is a perfect square a^2 of the form $n2^k - 7$ for some positive integer n. Then a is necessarily odd. We shall show how to produce a perfect square of the form $n'2^{k+1} - 7$ for some positive integer n'. If n is even, then $a^2 = (n/2)2^{k+1} - 7$ is of the required form. Suppose that n is odd. We wish to choose a positive integer m such that $(a+m)^2$ is of the desired form.
Consider $(a+m)^2 = a^2 + 2am + m^2 = -7 + n2^k + m(m+2a)$. If we choose $m = 2^{k-1}$, then $m(m+2a)$ is an odd multiple of 2^k. Consequently, $(a+m)^2$ is of the form $n'2^{k+1} - 7$ for some positive integer n'. Now the solution of this problem can be completed by induction on k.

1.3. Let p be the smallest positive integer such that $pa \equiv 0 \pmod{1995}$, i.e. $pa = 1995k$ for some positive integer k. Let $q = 1995/p$. Then q is an integer and it divides a. We claim that

$$S = \{ma + nb \pmod{1995} \mid m = 0, 1, \ldots, \ p-1, n = 0, 1, \ldots, q-1\}$$

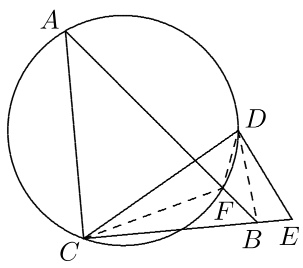
First note that there are $pq = 1995$ elements in the set on the right hand side. It suffices to prove that the elements are distinct. Suppose that $ma + nb \equiv m'a + n'b \pmod{1995}$. Then $(m-m')a + (n-n')b = 1995\ell$ for some integer ℓ. Since q divides 1995 and a, and q is relatively prime to b, we have q divides $(n-n')$. But $|n-n'| \le q-1$, so $n-n' = 0$. Consequently, $m = m'$. This completes the proof of the claim.

Consider the following sequence:

$$\underbrace{a, a, \ldots, a, b}_{p \text{ terms}}, \underbrace{-a, -a, \ldots, -a, b}_{p \text{ terms}}, \underbrace{a, a, \ldots, a, b}_{p \text{ terms}}, \ldots, \underbrace{(-1)^q a, (-1)^q a, \ldots, (-1)^q a, b}_{p \text{ terms}}$$

In this sequence, there are q blocks of $a, a, \ldots, a, b$ or $-a, -a, \ldots, -a, b$ making a total of $pq = 1995$ terms. For each $i = 1, 2 \ldots, 1995$, let s_i be the sum of the first i terms of this sequence. Then by the result above, $\{s_1, s_2 \ldots, s_{1995}\} = S$ and $s_{i+1} - s_i = \pm a$ or $\pm b$ (mod 1995).

2.1. Since $\angle CDF = \angle CAF = 45^o$, we have $\angle FDE = \angle CDE - \angle CDF = 45^o = \angle CDF$. Hence DF bisects $\angle CDE$. As $CB = CD$, we have $\angle CBD = \angle CDB$. Hence $\angle FBD = \angle CBD - 45^o = \angle CDB - 45^o = \angle FDB$. Therefore $FD = FB$. This shows that $\triangle BCF$ is congruent to $\triangle DCF$. Hence $\angle BCF = \angle DCF$ and CF bisects $\angle DCE$. Therefore F is the incentre of $\triangle CDE$.



2.2. Let $\mathbb{N}$ be the set of all natural numbers. Let $A = \{n^2 \mid n \in \mathbb{N}\}$. Let $\mathbb{N} \setminus A = \{n_1, n_2, n_3, \ldots\}$. Define f as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ n_{2i} & \text{if } n = n_{2i-1}, \quad i = 1, 2, \ldots \\ n_{2i-1}^2 & \text{if } n = n_{2i}, \quad i = 1, 2, \ldots \\ n_{2i}^{2^k} & \text{if } n = n_{2i-1}^{2^k}, \quad k = 1, 2, \ldots \\ n_{2i-1}^{2^{k+1}} & \text{if } n = n_{2i}^{2^k}, \quad k = 1, 2, \ldots \end{cases}$$

Then $f : \mathbb{N} \longrightarrow \mathbb{N}$ satisfies the requirement $f(f(n)) = n^2$.

(Note: The function above comes from the following consideration. First, $f(1)$ must be 1. Let $f(2) = 3$. Then $f(3) = 2^2$, $f(2^2) = 3^2$, $f(3^2) = 2^4$ etc.. Next, let $f(5) = 6$. Continuing as before, we have $f(6) = 5^2$, $f(5^2) = 6^2$, $f(6^2) = 5^4$ etc..)

2.3. Let $N = \{1, 2, ..., 1995\}$. Let q be an integer with $1 \leq q \leq m$. We shall prove the following statement *S(q)* by induction (on q):

S(q): There exists a subset I_q of N such that $\sum_{i \in I_q} n_i = q$.

$S(1)$ is true because one of the n_i's must be 1. Now assume that for some q with $1 \leq q < m$, $S(i)$ is true for $i \leq q$. Then $|I_q| \leq q$ and 1994.

If $n_i > q + 1$ for all $i \in N \setminus I_q$, then $\sum_{i \in N} n_i \geq q + (q + 2)(1995 - |I_q|) = (1996 - |I_q|)q + 2(1995 - |I_q|) \geq 2q + 2(1995 - |I_q|) \geq 2q + 2(1995 - q) = 3990$, which is a contradiction. Hence, there exists $j \in N \setminus I_q$ such that $n_j \leq q+1$. Let $a = \min\{n_i : i \notin I_q\}$. Then $a \leq q+1$ and $a - 1 \leq q$. Thus $S(a - 1)$ is true. By the choice of a, there exists $J \subseteq I_q$ such that $a - 1 = \sum_{i \in J} n_i$. Therefore, $q + 1 = q + a - (a - 1) = \sum_{i \in I_q \setminus J} n_i + a$. Thus, $S(q + 1)$ is true.

This problem appears in the American Mathematical Monthly with 1995 replaced by k and 3990 replaced by $2k$. The proof above works for the general case too. See (The American Mathematical Monthly, Vol.105, No.3, March 1998, pg 273-274.)

1996/97

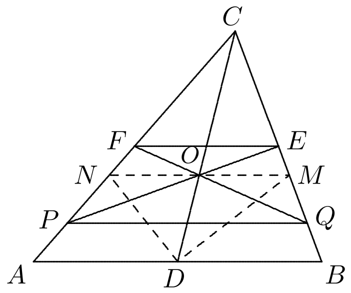
1.1. Since DM and DN are angle bisectors of $\angle BDC$ and $\angle ADC$ respectively, by Stewart's theorem, we have

$$\frac{BM}{MC} = \frac{DB}{DC} \quad \text{and} \quad \frac{AN}{NC} = \frac{AD}{DC}.$$

As $AD = DB$, we have $\dfrac{BM}{MC} = \dfrac{AN}{NC}$.

Hence $NM//AB$ and $\triangle ABC \sim \triangle NMC$.

Therefore $\dfrac{AB}{NM} = \dfrac{AC}{NC} = \dfrac{BC}{MC}$.



Since $\dfrac{BM}{MC} = \dfrac{DB}{DC}$, we have $\dfrac{DB+DC}{DC} = \dfrac{BM+MC}{MC} = \dfrac{BC}{MC} = \dfrac{AB}{NM}$.

On the other hand, $FE = \frac{1}{2}AB = DB$. Therefore, $\dfrac{FE+DC}{DC} = \dfrac{2FE}{NM}$.

Consequently, $\dfrac{1}{FE} + \dfrac{1}{DC} = \dfrac{2}{NM}$.

Applying Menelaus's Theorem to $\triangle CMN$ for the lines EP and FQ and using the fact that $OM = ON$, we have

$$\frac{CP}{PN} = \frac{OM}{ON} \cdot \frac{CE}{ME} = \frac{CE}{ME} \quad \text{and} \quad \frac{CQ}{QM} = \frac{ON}{OM} \cdot \frac{FC}{FN} = \frac{FC}{FN}.$$

Since $FE//AB//NM$, we have $\dfrac{CE}{ME} = \dfrac{FC}{FN}$. Therefore $\dfrac{CQ}{QM} = \dfrac{CP}{PN}$ so that $FE//PQ$.

Hence $PQEF$ is a trapezoid and O is the intersection point of its two diagonals.

From this, it follows that $\dfrac{1}{FE} + \dfrac{1}{PQ} = \dfrac{2}{NM}$. Consequently, $PQ = DC$.

1.2. It can be shown that a_n satisfies the recurrence relation: $a_n = 2a_{n-1} + 2a_{n-2}$ with $a_1 = 3$ and $a_2 = 8$. Solving this difference equation gives

$$a_n = (\frac{1}{2} + \frac{1}{\sqrt{3}})(1+\sqrt{3})^n + (-1)^{n+1}(\frac{1}{\sqrt{3}} - \frac{1}{2})(\sqrt{3}-1)^n.$$

Next we shall show that $(\frac{1}{\sqrt{3}} - \frac{1}{2})(\sqrt{3}-1)^n < 0.5$ for $n \geq 1$. This is because

$$\text{for } n \geq 1, \quad 0 < (\frac{1}{\sqrt{3}} - \frac{1}{2})(\sqrt{3}-1)^n \leq (\frac{1}{\sqrt{3}} - \frac{1}{2})(\sqrt{3}-1) < (1 - \frac{1}{2})(2-1) = 0.5.$$

Thus $a_n = (\frac{1}{2} + \frac{1}{\sqrt{3}})(1+\sqrt{3})^n$ rounded off to the nearest integer.

1.3. **1st solution**

Let $x \in \mathbb{R}$. By letting $x = y + f(0)$, we obtain

$$f(f(x)) = f(f(y + f(0))) = f(0 + f(y)) = y + f(0) = x.$$

Hence for any $t_1, t_2 \in \mathbb{R}$, $f(t_1 + t_2) = f(t_1 + f(f(t_2))) = f(t_1) + f(t_2)$.
Next, consider any positive integer m such that $m \neq -f(x)$. We have

$$\frac{f(m + f(x))}{m + f(x)} = \frac{x + f(m)}{m + f(x)} = \frac{x + mf(1)}{m + f(x)}.$$

Since the set $\{\frac{f(t)}{t} \mid t \neq 0\}$ is finite, there exist distinct positive integers m_1, m_2 with $m_1, m_2 \neq -f(x)$ such that

$$\frac{f(m_1 + f(x))}{m_1 + f(x)} = \frac{f(m_2 + f(x))}{m_2 + f(x)}.$$

Hence $\frac{x + m_1 f(1)}{m_1 + f(x)} = \frac{x + m_2 f(1)}{m_2 + f(x)}$. From this, we have $f(x)f(1) = x$.

By letting $x = 1$, we obtain $[f(1)]^2 = 1$ so that $f(1) = \pm 1$. Consequently, $f(x) = \pm x$. Also the functions $f(x) = x$ and $f(x) = -x$ clearly satisfy the two given conditions.

2nd solution
(i) First we prove that $f(0) = 0$. Putting $x = 0 = y$, we have $f(f(0)) = f(0)$. If $f(0) = a$, then $f(0) = f(f(0)) = f(a)$. Thus $a + f(0) = f(0 + f(a)) = f(f(0)) = f(0)$, whence $a = 0$.
(ii) Putting $x = 0$, we have $f(f(y) = y$ for all y.
(iii) We will prove that $f(x) = \pm x$ for all x.
Suppose for some p, $f(p) = cp$ for some constant $c \neq \pm 1$. Then $f(p + f(p)) = p + f(p)$. Let $q = p + f(p)$. Then $q \neq 0$ and $f(q) = q$. Thus $f(q + f(q)) = q + f(q)$ and $f(2q) = 2q$. Inductively we have $f(nq) = nq$ for any positive integer n. Now $f(nq + f(p)) = p + f(nq)$. So $f(nq + cp) = p + nq$. Thus $f(nq + cp)/(nq + cp) = 1 - (c-1)p/(nq + cp)$. Since $c - 1 \neq 0$ and there are infinitely many choices for n so that $nq + cp \neq 0$, this gives an infinite number of members in the set $\{f(x)/x\}$ contradicting the second condition. Thus $c = \pm 1$.
(iv) For $f(p) = p$, we will prove that $f(x) = x$ for all x.
If f$(-p) = p$, then $-p = f(f(-p)) = f(p) = p$ which is impossible. Thus $f(-p) = -p$.
Suppose there exists r such that $f(r) = -r$. Then $f(r + f(p)) = p + f(r)$, i.e., $f(r + p) = p - r$. Therefore $f(r + p)/(r + p) = (p - r)/(r + p) \neq \pm 1$. (Note that the denominator is not zero.)
(v) From the above we conclude that either $f(x) = x$ for all x or $f(x) = -x$ for all x. Clearly these functions satisfy the two given conditions. Thus these are the only two functions required.

2.1. Let a, b, c, d represent the numbers at any stage subsequent to the initial one. Then $a + b + c + d = 0$ so that $d = -(a + b + c)$. It follows that

$$\begin{aligned} bc - ad &= bc + a(a + b + c) = (a + b)(a + c), \\ ac - bd &= ac + b(a + b + c) = (a + b)(b + c), \\ ab - cd &= ab + c(a + b + c) = (a + c)(b + c). \end{aligned}$$

Hence, $|(bc - ad)(ac - bd)(ab - cd)| = (a + b)^2(a + c)^2(b + c)^2$.

Therefore the product of the three quantities $|bc - ad|, |ac - bd|, |ab - cd|$ is the square of an integer. However the product of three primes cannot be the square of an integer, so the answer to the question is "NO".

2.2. $\binom{n-i+1}{i}$ is equal to the number of i-subsets of the set $S = \{1, 2, \ldots, n\}$ containing no consecutive integers. Hence the required sum is just the number a_n of subsets of S containing no consecutive integers. It can be shown easily that a_n satisfies the recurrence relation: $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 1$ and $a_1 = 2$. This can also be derived from the identity:

$$\binom{n-i+1}{i} = \binom{(n-1)-i+1}{i} + \binom{(n-2)-(i-1)+1}{i-1}.$$

From this, we obtain

$$\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i+1}{i} = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

2.3. We shall prove by induction on k that

$$\frac{n+1}{2n-k+2} < a_k < \frac{n}{2n-k} \quad \text{for} \quad k = 1, 2, \ldots, n.$$

For $k = 1$, we have

$$a_1 = a_0 + \frac{1}{n}a_0^2 = \frac{2n+1}{4n},$$

Hence

$$\frac{n+1}{2n+1} < a_1 < \frac{n}{2n-1},$$

so the induction hypothesis is true for $k = 1$.
Now suppose the induction hypothesis is true for $k = r < n$, then

$$a_{r+1} = a_r + \frac{1}{n}a_r^2 = a_r\left(1 + \frac{1}{n}a_r\right).$$

Hence we have

$$\begin{aligned} a_{r+1} &> \frac{n+1}{2n-r+2}\left(1 + \frac{1}{n} \cdot \frac{n+1}{2n-r+2}\right) \\ &> \frac{n+1}{2n-r+1} = \frac{n+1}{2n-(r+1)+2}. \end{aligned}$$

On the other hand,

$$a_{r+1} < \frac{n}{2n-r}\left(1 + \frac{1}{n} \cdot \frac{n}{2n-r}\right) = \frac{n(2n-r+1)}{(2n-r)^2} < \frac{n}{2n-(r+1)},$$

since $(2n-r)^2 > (2n-r+1)(2n-(r+1))$. Hence the induction hypothesis is true for $k = r+1$. This completes the induction step.
When $k = n$, we get

$$1 - \frac{1}{n} < 1 - \frac{1}{n+2} = \frac{n+1}{n+2} < a_n < \frac{n}{2n-n} = 1,$$

the required inequality.

1997/98

1.1. Let $AC = a$, $CE = b$, $AE = c$. Applying the Ptolemy's Theorem[1] for the quadrilateral $ACEF$ we get
$$AC \cdot EF + CE \cdot AF \geq AE \cdot CF.$$
Since $EF = AF$, it implies $\frac{FA}{FC} \geq \frac{c}{a+b}$. Similarly, $\frac{DE}{DA} \geq \frac{b}{c+a}$ and $\frac{BC}{BE} \geq \frac{a}{b+c}$. It follows that
$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}. \tag{1}$$
The last inequality is well known[2]. For equality to occur, we need equality to occur at every step of (1) and we need an equality each time Ptolemy's Theorem is used. The latter happens when the quadrilateral $ACEF$, $ABCE$, $ACDE$ are cyclic, that is, $ABCDEF$ is a cyclic hexagon. Also for the equality in (1) to occur, we need $a = b = c$. Hence equality occurs if and only if the hexagon is regular.

1.2. We will prove the statement by induction on n. It obviously holds for $n = 2$. Assume that $n > 2$ and that the statement is true for any integer less than n. We distinguish two cases.
Case **1**. There are no i and j such that $A_i \cup A_j = S$ and $|A_i \cap A_j| = 1$.
Let x be an arbitrary element in S. The number of sets A_i not containing x is at most $2^{n-2} - 1$ by the induction hypothesis. The number of subsets of S containing x is 2^{n-1}. At most half of these appear as a set A_i, since if $x \in A_i$, then there is no j such that $A_j = (S - A_i) \cup \{x\}$ for otherwise $|A_i \cap A_j| = 1$. Thus the number of sets A_i is at most $2^{n-2} - 1 + 2^{n-2} = 2^{n-1} - 1$.
Case **2**. There is an element $x \in S$ such that $A_1 \cup A_2 = S$ and $A_1 \cap A_2 = \{x\}$.
Let $|A_1| = r + 1$ and $|A_2| = s + 1$. Then $r + s = n - 1$. The number of sets A_i such that $A_i \subseteq A_1$ is at most $2^r - 1$ by the induction hypothesis. Similarly the number of sets A_i such that $A_i \subseteq A_2$ is at most $2^s - 1$.
If A_i is not a subset of A_1 and A_2, then $A_1 \cap A_i \neq \emptyset$, $A_2 \cap A_i \neq \emptyset$. Since $A_1 \cap A_2 \neq \emptyset$, we have $A_1 \cap A_2 \cap A_i \neq \emptyset$. Thus $A_1 \cap A_2 \cap A_i = \{x\}$. Thus $A_i = \{x\} \cup (A_i - A_1) \cup (A_i - A_2)$, and since the nonempty sets $A_i - A_1$ and $A_i - A_2$ can be chosen in $2^s - 1$ and $2^r - 1$ ways, respectively, the number of these sets is at most $(2^s - 1)(2^r - 1)$. Adding up these partial results we obtain the result that the number of A_i's is at most $2^{n-1} - 1$.

1.3. **1st solution**
Note that for any a, b, we have $(a - b)| \pm (F(a) - F(b))$. Thus 1998 divides $F(1998) - F(0)$, whence $F(1998) = F(0)$ as $|F(1998) - F(0)| \leq 1997$. Also we have $4 = 1998 - 1994$ divides $F(1994) - F(1998) = F(1994) - F(0)$, and $1994|(F(1994) - F(0))$. Thus LCM $(4, 1994) = 3988$ divides $F(1994) - F(0)$ which implies $F(1994) = F(0)$. By reversing the role of 4 and 1998, we have $F(4) = F(0)$. By considering 5 and 1993, we also have $F(1993) = F(5) = F(0)$. Then for any a, $1 \leq a \leq 1997$, we have $(x - a)|F(0) - F(a)$ for $x = 4, 5, 1993, 1994$. The least common multiplier of the 4 numbers $x - a$ is larger than 1998. Thus $F(a) = F(0)$.

2nd solution
We shall prove that the statement holds for any integer $k \geq 4$, not just $k = 1998$. Consider any polynomial $F(x)$ with integer coefficients satisfying the given inequality $0 \leq F(c) \leq k$ for every $c \in \{0, 1, \ldots, k + 1\}$. Note that $F(k + 1) = F(0)$ because $F(k + 1) - F(0)$ is a multiple of $k + 1$ not exceeding k in absolute value. Hence
$$F(x) - F(0) = x(x - k - 1)G(x)$$

where $G(x)$ is a polynomial with integer coefficients. Thus

$$k \geq |F(c) - F(0)| = c(k+1-c)|G(c)| \quad \text{for each } c \in \{1, 2 \ldots, k\}. \tag{2}$$

The inequality $c(k+1-c) > k$ holds for each $c \in \{1, 2, \ldots, k-1\}$ which is not an empty set if $k \geq 3$. Thus for any c in this set, $|G(c)| < 1$. Since $G(c)$ is an integer, $G(c) = 0$. Thus $2, 3, \ldots, k-1$ are roots of $G(x)$, which yields

$$F(x) - F(0) = x(x-2)(x-3)\cdots(x-k+1)(x-k-1)H(x). \tag{3}$$

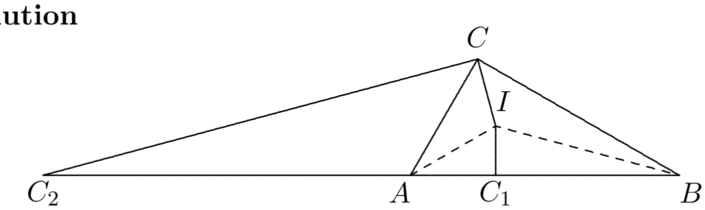
We still need to prove that $H(1) = H(k) = 0$. For both $c = 1$ and $c = k$, (3) implies that

$$k \geq |F(c) - F(0)| = (k-2)! \cdot k \cdot |H(c)|.$$

Now $(k-2)! > 1$ since $k \geq 4$. Therefore $|H(c)| < 1$ and hence $H(c) = 0$.
For $k = 1, 2, 3$ we have the following counterexamples:

$$\begin{array}{ll} F(x) = x(2-x) & \text{for } k = 1 \\ F(x) = x(3-x) & \text{for } k = 2 \\ F(x) = x(4-x)(x-2)^2 & \text{for } k = 3 \end{array}$$

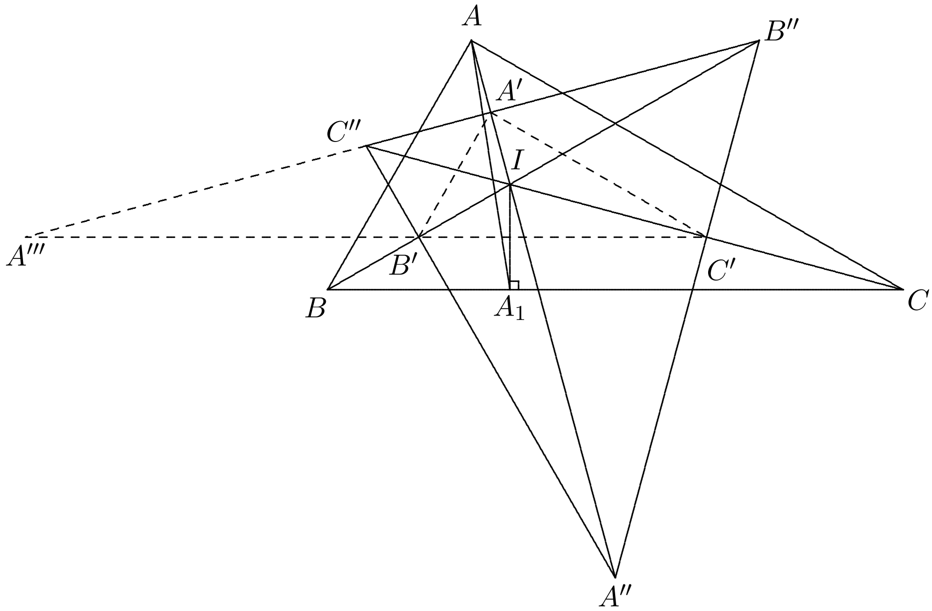
2.1. **1st solution**



Let the line perpendicular to CI and passing through C meet AB at C_2. By analogy, we denote the points A_2 and B_2. It's obvious that the centres of the circumcircles of AIA_1, BIB_1 and CIC_1 are the middle points of A_2I, B_2I and C_2I, respectively. So it's sufficient to prove that A_2, B_2 and C_2 are collinear. Let's note that CC_2 is the exterior bisector of $\angle ACB$, and so $\frac{C_2A}{C_2B} = \frac{CA}{CB}$. By analogy $\frac{B_2A}{B_2C} = \frac{BA}{BC}$ and $\frac{A_2B}{A_2C} = \frac{AB}{AC}$. Thus $\frac{C_2A}{C_2B}\frac{B_2C}{B_2A}\frac{A_2B}{A_2C} = \frac{CA}{CB}\frac{BC}{BA}\frac{AB}{AC} = 1$ and by Menelaus' Theorem[3], the points A_2, B_2 and C_2 are collinear.

2nd solution

Let A', B', C' be the midpoints of AI, BI, CI, respectively. Let the perpendicular bisectors of AI and BI meet at C''. A'' and B'' are similarly defined.



Then the circumcentre A''' of AIA_1 is the intersection of $B''C''$ with $B'C'$. Likewise the circumcentre B''' of BIB_1 is the intersection of $A''C''$ with $A'C'$ and the circumcentre C''' of CIC_1 is the intersection of $A''B''$ with $A'B'$.
First we note that the circumcentre of AIB lies on the line CI. To prove this, let the circumcircle of AIB meet CI at another point X. Then $\angle XAB = \angle XIB = \frac{1}{2}(\angle B + \angle C)$. Thus $\angle XAI = \angle XAB + \angle BAI = 90°$. Thus XI is a diameter and the circumcentre which is C'' is on the line CI. Similarly, A'' is on AI and B'' is on BI.
Now we consider the triangles $A'B'C'$ and $A''B''C''$. The lines $A'A''$, $B'B''$, and $C'C''$ are concurrent (at I), thus by Desargues' Theorem[4], the three points, namely, the intersections of $B''C''$ with $B'C'$, $A''C''$ with $A'C'$ and $A''B''$ with $A'B'$ are collinear.

3rd solution (By inversion)
Let c be the incircle of $\triangle ABC$ of radius r. The image of a point X under the inversion about c is the point X^* such that $IX \cdot IX^* = r^2$. Inversion about a circle c has the following properties:

(a) If X lies on c, then $X^* = X$.
(b) $I^* = \infty$.
(c) If s is a circle intersecting c at two points P, Q and s passes through I, then s^* is a straight line passing through P and Q.

Now $A^* = A_o$, where A_o is the midpoint of B_1C_1. Also, $A_1{}^* = A_1$ and $I^* = \infty$. Hence, the inversion of the circumcircle of $\triangle AIA_1$ is the line A_1A_o. Similarly, the inversion of the circumcircle of $\triangle BIB_1$ is the line B_1B_o and the inversion of the circumcircle of $\triangle CIC_1$ is the line C_1C_o, where B_o is the midpoint of C_1A_1 and C_o is the midpoint of A_1B_1. Note that the 3 medians A_1A_o, B_1B_o, C_1C_o of $\triangle A_1B_1C_1$ are concurrent. Furthermore, they meet at ∞. This means that the circumcircles under consideration pass through two points. (one of them is I.) Thus they are coaxial and hence their centres are collinear.

2.2. **1st solution**
We need to prove that

$$\sqrt{\sum_{k=1}^{n} a_k} \le \sum_{k=1}^{n-1} \sqrt{k}(\sqrt{a_k} - \sqrt{a_{k+1}}) + \sqrt{na_n}.$$

We prove this by induction on n. For $n = 1$ the void sum has value zero and the result is clear. Assume that the result holds for a certain $n \ge 1$. Consider $a_1 \ge \cdots \ge a_{n+1} \ge a_{n+2} = 0$. Write $S = \sum_{k=1}^{n} a_k$ and $b = a_{n+1}$. It suffices to prove that

$$\sqrt{S+b} - \sqrt{S} \le -\sqrt{nb} + \sqrt{(n+1)b}.$$

This holds trivially when $b = 0$. And if $b > 0$, division by $\sqrt{b}$ takes it into the form

$$\sqrt{U+1} - \sqrt{U} \le \sqrt{n+1} - \sqrt{n},$$

where $U = S/b$; equivalently:

$$\frac{1}{\sqrt{U+1} + \sqrt{U}} \le \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Since $b = a_{n+1} \le S/n$, we have $U \ge n$, whence the last inequality is true and the proof is complete.

2nd solution

Set $x_k = \sqrt{a_k} - \sqrt{a_{k+1}}$, for $k = 1, \dots, n$. Then

$$a_1 = (x_1 + \cdots + x_n)^2, \quad a_2 = (x_2 + \cdots + x_n)^2, \dots, a_n = x_n^2.$$

Expanding the squares we obtain

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k x_k^2 + 2 \sum_{1 \le k < \ell \le n} k x_k x_\ell. \tag{3}$$

Note that the coefficient of $x_k x_\ell$ (where $k < \ell$) in the last sum is equal to k. The square of the right-hand side of the asserted inequality is equal to

$$\Big(\sum_{k=1}^{n} \sqrt{k} x_k\Big)^2 = \sum_{k=1}^{n} k x_k^2 + 2 \sum_{1 \le k < l \le n} \sqrt{k\ell} x_k x_\ell. \tag{4}$$

And since the value of (3) is obviously not greater than the value of (4), the result follows.

3rd solution Let $c_k = \sqrt{k} - \sqrt{k-1}$, then the inequality can be transformed to

$$\sqrt{\sum_{k=1}^{n} a_k} \le \sum_{k=1}^{n} \sqrt{a_k} c_k.$$

By squaring both sides, this is in turn equivalent to

$$\sum_{k=2}^{n} a_k (c_k^2 - 1) + \sum_{0 \le i < j \le n} 2\sqrt{a_i a_j} c_i c_j \ge 0.$$

Note that $c_i c_j = \sqrt{ij} - \sqrt{i(j-1)} - \sqrt{(i-1)j} + \sqrt{(i-1)(j-1)}$. Thus for $k = 3, \dots, n$,

$$\begin{aligned}
\sum_{i=1}^{k-1} 2\sqrt{a_i a_k} c_i c_k &= \sum_{i=1}^{k-2} 2\big(\sqrt{ik} - \sqrt{i(k-1)}\big)\big(\sqrt{a_i a_k} - \sqrt{a_{i+1} a_k}\big) \\
&= +2\sqrt{a_{k-1} a_k}\big(\sqrt{k(k-1)} - (k-1)\big) \\
&\ge 2\sqrt{a_{k-1} a_k}\big(\sqrt{k(k-1)} - (k-1)\big) = \sqrt{a_{k-1} a_k}(1 - c_k^2).
\end{aligned}$$

Also $2\sqrt{a_1 a_2} c_1 c_2 = \sqrt{a_1 a_2}(1 - c_2^2)$. Hence

$$\begin{aligned}
&\sum_{k=2}^{n} a_k (c_k^2 - 1) + \sum_{0 \le i < j \le n} 2\sqrt{a_i a_j} c_i c_j \\
&\quad \ge \sum_{k=2}^{n} a_k (c_k^2 - 1) + \sum_{k=2}^{n} \sqrt{a_{k-1} a_k}(1 - c_k^2) \\
&\quad = \sum_{k=2}^{n} (1 - c_k^2)\big(\sqrt{a_{k-1} a_k} - a_k\big) \ge 0.
\end{aligned}$$

since $\sqrt{a_{k-1} a_k} - a_k \ge 0$ and $1 - c_k^2 \ge 0$. This completes the proof.
From solutions 2 and 3, we can conclude that equality holds if and only if there exists an index m such that $a_1 = \cdots = a_m$ and $a_k = 0$ for $k > m$.

2.3. **1st solution**
We prove by induction on h, the common difference of the progression. If $h = 1$, there is nothing to prove. Fix $h > 1$ and assume that the statement is true for progressions whose common difference is less than h. Consider an arithmetic progression with first term

a, and common difference h such that both x^2 and y^3 are terms in the progression. Let $d = \gcd(a, h)$. Write $h = de$. If an integer n satisfies $n \equiv a \pmod h$ and $n \geq a$, then n is a term in the progression. Thus it suffices to prove that there is a z satisfying $z^6 \equiv a \pmod h$ as this implies $(z + kh)^6 \equiv a \pmod h$ for any positive integer k and one can always choose a large k so that $(z + kh)^6 \geq a$.

Case **1**. $\gcd(d, e) = 1$: We have $x^2 \equiv a \equiv y^3 \pmod h$, hence also $\pmod e$. The number e is coprime to a, hence to x and y as well. So there exists an integer t such that $ty \equiv x \pmod e$. Consequently $(ty)^6 \equiv x^6 \pmod e$, which can be rewritten as $t^6a^2 \equiv a^3 \pmod e$. Dividing by a^2 (which is legitimate because $\gcd(a, e) = 1$), we obtain $t^6 \equiv a \pmod e$. As $\gcd(d, e) = 1$, it follows that $t + ke \equiv 0 \pmod d$ for some integer k. Thus

$$(t + ke)^6 \equiv 0 \equiv a \pmod d.$$

Since $t^6 \equiv a \pmod e$, we get from the Binomial Formula

$$(t + ke)^6 \equiv a \pmod e.$$

And since d and e are coprime and $h = de$, the latter two equations imply

$$(t + ke)^6 \equiv a \pmod h.$$

Case **2**. $\gcd(d, e) > 1$. Let p be a prime divisor of d and e. Assume that p^α is the greatest power of p dividing a and p^β is the greatest power of p dividing h. Recalling that $h = de$ with e being coprime to a, we see that $\beta > \alpha \geq 1$. If follows that for each term of the progression $(a + ih : i = 0, 1, \ldots)$, the greatest power of p which divides it is p^α. Since x^2 and y^3 are in the progression, α must be divisible by 2 and 3. So $\alpha = 6\gamma$ for some integer γ; hence $\alpha \geq 6$.
The progression $(p^{-6}(a + ih) : i = 1, 2, \ldots)$ with common difference $h/p^6 < h$ has integer terms and contains the numbers $(x/p^3)^2$ and $(y/p^2)^3$. By the induction hypothesis it contains a term z^6 for some integer z. Thus $(pz)^6$ is a term in the original progression. This completes the induction.

2nd solution
We use the same notation as in the first solution.
The assertion is proved by induction on h. The case $d = 1$ is trivially true.
(a) $\gcd(a, h) = 1$. (a^{-1} exists mod h.) In this case, we have $(y/x)^6 \equiv a \pmod h$.
(b) $\gcd(a, h) = r > 1$. Pick a prime p dividing r and let α be the largest positive integer such that p^α divides r. If $\alpha \geq 6$, then

$$(\frac{x}{p^2})^3 \equiv \frac{a}{p^6}, \quad (\frac{y}{p^3})^2 \equiv \frac{a}{p^6} \qquad (\text{mod } \frac{d}{p^6}).$$

By induction hypothesis, there exists z such that $z^6 \equiv \frac{a}{p^6}$ $(\text{mod } \frac{d}{p^6})$. Then $(zp)^6 \equiv a \pmod h$. So we suppose $0 < \alpha < 6$. From $x^3 \equiv a,\ y^2 \equiv a \pmod h$, we have

$$\frac{x^3}{p^\alpha} \equiv \frac{a}{p^\alpha}, \quad \frac{y^2}{p^\alpha} \equiv \frac{a}{p^\alpha} \qquad (\text{mod } \frac{d}{p^\alpha}). \tag{$*$}$$

(i) $\gcd(p, \frac{h}{p^\alpha}) = 1$. ($p^{-1}$ exists mod $\frac{d}{p^\alpha}$.) Multiply both sides of (*) by $p^{\alpha-6}$. We have

$$(\frac{x}{p^2})^3 \equiv \frac{a}{p^6}, \quad (\frac{y}{p^3})^2 \equiv \frac{a}{p^6} \qquad (\text{mod } \frac{d}{p^\alpha}).$$

By induction hypothesis, there exists z such that $z^6 \equiv \frac{a}{p^6} \pmod{\frac{d}{p^\alpha}}$. Write $a = p^\alpha a'$, then there is an integer m such that

$$(pz)^6 - p^\alpha a' = m\frac{h}{p^\alpha}.$$

Since $\alpha < 6$, p^α divides the left hand side of the equation. Thus it also divides m, whence $(pz)^6 \equiv p^\alpha a' = a \pmod{h}$.
(ii) $\gcd(p, \frac{h}{p^\alpha}) = p$. Then p^α is the largest power of p dividing a. Furthermore, α is a multiple of 3. To see this write $x = p^\beta x'$, where p does not divide x' and let $x = a + kh$ for some positive integer k. Then $p^{3\beta}x'^3 = x^3 = a + kh = p^\alpha(a' + pkh')$ for some integer a', h' with $\gcd(a', p) = 1$. Consequently, $\alpha = 3\beta$. Similarly, α is a multiple of 2. Therefore, $\alpha \geq 6$, and this case does not arise.

Footnotes
1. **Ptolemy's Theorem.** For any quadrilateral $ABCD$, we have

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD$$

and equality occurs if and only if $ABCD$ is cyclic.
2. **Proof of the inequality.** Let $x = a + b$, $y = a + c$, $z = b + c$, then

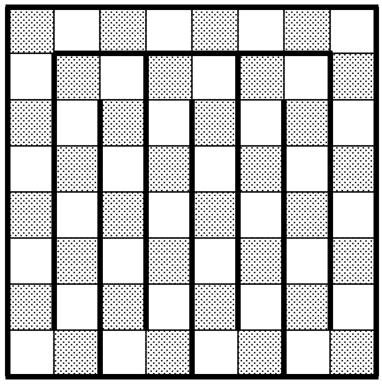
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x} + \frac{x}{z} + \frac{z}{x} + \frac{y}{z} + \frac{z}{y} - 3\right) \geq \frac{3}{2}.$$

3. **Menelaus' Theorem.** Three points X, Y and Z on the sides BC, CA, and AB (suitably extended) of triangle ABC are collinear if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

4. **Desargues' Theorem.** Given any pair of triangles ABC and $A'B'C'$, the following are equivalent: (i) The lines AA', BB' and CC' are concurrent. (ii) The points of intersection of AB with $A'B'$, AC with $A'C'$, BC with $B'C'$ are collinear.

1988/89
1.1. Suppose p is an integer such that $p^3 - mp^2 + mp - (m^2 + 1) = 0$. It follows that $(p^2 + m)(p - m) = 1$. Since p and m are integers, we have either
(1) $p^2 + m = p - m = -1$, or
(2) $p^2 + m = p - m = 1$.
In case (1), we have $m = p + 1$, and so $p^2 + p + 1 = -1$ or $p^2 + p + 2 = 0$ which has no real solution.
In case (2), we have $m = p - 1$, and so $p^2 + p - 1 = 1$ or $p^2 + p - 2 = 0$ which has the solutions $p = -2$ and 1.
Hence, $m = -3$ and 0 are the integer values of m for which the given equation has an integer solution.



1.2. It is only necessary to partition the checkerboard into a closed path one square wide. One way to do this is shown in the diagram. The squares lie with alternating colours along the closed path. The removal of two squares of opposite colours from any two positions along the path will cut the path into two open-ended segments (or one segment if the removed squares are adjacent on the path).
Each segment must consist of an even number of squares, so each segment must be completely covered by dominoes.

1.3. **1st solution**
More generally, let S_n be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ such that

(i) $x_i = \pm 1$ for $i = 1, \ldots, n$,

(ii) $0 \le x_1 + x_2 + \cdots + x_r < 4$, for $r = 1, 2, \ldots, n-1$,

(iii) $x_1 + x_2 + \cdots + x_n = 4$.

Also let S'_n be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ such that

(i) $x_i = \pm 1$ for $i = 1, \ldots, n$,

(ii) $-2 \le x_1 + x_2 + \cdots + x_r < 2$, for $r = 1, 2, \ldots, n-1$,

(iii) $x_1 + x_2 + \cdots + x_n = 2$.

Let $|S_n| = a_n$ and $|S'_n| = b_n$. First note that $a_n = 0$ if n is odd. Hence we consider only even values of n.

Let $(x_1, x_2 \ldots, x_{2k})$ be an element in S_{2k}. Then $(x_1, x_2) = (1, 1)$ or $(1, -1)$. If $(x_1, x_2) = (1, 1)$, then $(x_3, x_4 \ldots, x_{2k})$ is an element of S'_{2k-2}. If $(x_1, x_2) = (1, -1)$, then $(x_3, x_4 \ldots, x_{2k})$ is an element of S_{2k-2}. Conversely, if each element of S'_{2k-2} is augmented at the beginning by two ones, it gives rise to an element of S_{2k}. Similarly, if each element of S_{2k-2} is augmented at the beginning by 1 and -1, it gives rise to an element of S_{2k}. This shows that $a_{2k} = b_{2k-2} + a_{2k-2}$.
Next consider an element $(x_1, x_2 \ldots, x_{2k})$ in S'_{2k}. Then $(x_1, x_2) = (1, -1)$, $(-1, 1)$ or $(-1, -1)$. Hence, $(x_3, x_4 \ldots, x_{2k})$ is an element of S'_{2k-2} if $(x_1, x_2) = (1, -1)$ or $(-1, 1)$, and it is an element of S_{2k-2} if $(x_1, x_2) = (-1, -1)$. By augmenting each element of S'_{2k-2} at the beginning by either 1,-1 or -1,1, it gives rise to an element of S'_{2k}. Similarly, by augmenting each element of S_{2k-2} at the beginning by $-1, -1$, we get an element of S'_{2k}. This shows that $b_{2k} = a_{2k-2} + 2b_{2k-2}$.
By eliminating the b_{2k}'s in the above two difference equations, we have $a_{2k} - 3a_{2k-2} + a_{2k-4} = 0$. The initial conditions are $a_2 = 0$ and $a_4 = 1$. By solving this difference equation, it gives

$$a_{2k} = \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^{2k-2} - \frac{1}{\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^{2k-2}.$$

2nd solution
Let S''_n be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ such that

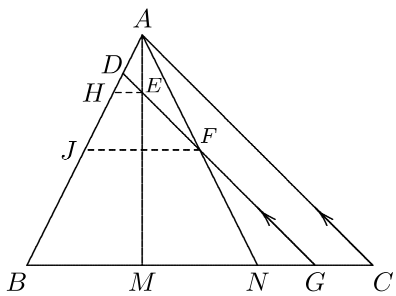
(i) $x_i = \pm 1$ for $i = 1, \ldots, n$,
(ii) $0 \le x_1 + x_2 + \cdots + x_r < 4$, for $r = 1, 2, \ldots, n-1$,
(iii) $x_1 + x_2 + \cdots + x_n = 2$.

Let $|S''_n| = c_n$. Again, we only have to consider even values of n. Note that each sequence in S_{2k} must end in two "1"s. By dropping these two ones, we obtain a sequence in S''_{2k-2}. Conversely, each sequence in S''_{2k-2} can be augmented at the end by two "1"s to get a sequence in S_{2k}. Hence $a_{2k} = c_{2k-2}$.
Let's examine the end terms of each sequence in S''_{2k}. The last three terms of any sequence in S''_{2k} are as follow:

$$-1\ 1\ 1,\quad 1\ -1\ 1,\quad 1\ 1\ -1,\quad -1\ -1\ 1,\quad -1\ 1\ -1. \quad (*)$$

For the first three cases, one can replace them by a single "1" to get a member of S''_{2k-2}. For the last two cases, one can drop the last two terms to get a member of S''_{2k-2}. Conversely, for any sequence in S''_{2k-2}, which ends in a "1", one can replace the "1" by any one of the first three endings in (*) to get a sequence in S''_{2k}.
Let the number of sequences in S''_{2k-2} which end in a "1" be x. Let the number of sequences in S''_{2k-2} which end in a "-1" be y. In the latter case, observe that if this last "-1" is replaced by a "1", then a sequence in S_{2k-2} is obtained. Hence, $y = a_{2k-2}$.
Similarly, for any sequence in S''_{2k-2} which ends in a "-1", it can be replaced by any one of the last two endings in (*) to get a sequence in S''_{2k}.
Therefore, $c_{2k} = 3x + 2y = 3(x+y) - y = 3c_{2k-2} - y$. That is $a_{2k+2} = 3a_{2k} - a_{2k-2}$. This is the same difference equation in solution 1.

2.1. Let H and J be the points on AB such that HE and JF are parallel to BC.



Then $\dfrac{GC}{NC} = \dfrac{FA}{NA} = \dfrac{JF}{BN} = \dfrac{JF}{2NC}$.

Hence $JF = 2GC$.

Also $\dfrac{EH}{MB} = \dfrac{AE}{AM} = \dfrac{GC}{MC} = \dfrac{GC}{2MB}$. This shows that $GC = 2EH$. Therefore,

$JF = 4EH$. As $\triangle DEH$ is similar to $\triangle DFJ$, we have $\dfrac{DF}{DE} = \dfrac{FJ}{EH} = 4$.

Consequently, $EF = DF - DE = 4DE - DE = 3DE$.

2.2. Let $r = x - \lfloor x \rfloor$. Write $\lfloor x \rfloor = lp + q$, where $l, q \in \mathbb{Z}$ and $0 \le q < |p|$. Hence $x = lp + q + r$. Now

$$\lfloor \frac{x-p}{p} \rfloor + \lfloor \frac{-x-1}{p} \rfloor = \lfloor \frac{q+r}{p} \rfloor + \lfloor \frac{-q-r-1}{p} \rfloor - 1.$$

So it suffices to find the value of the expression $A \equiv \lfloor \dfrac{q+r}{p} \rfloor + \lfloor \dfrac{-q-r-1}{p} \rfloor$.

(i) ($p > 0$.)

Then $\lfloor\frac{q+r}{p}\rfloor = 0$ and $\lfloor\frac{-q-r-1}{p}\rfloor = \begin{cases} -2 & \text{if } q = p-1 \text{ and } r > 0 \\ -1 & \text{otherwise} \end{cases}$.

Hence, $A = \begin{cases} -2 & \text{if } q = p-1 \text{ and } r > 0 \\ -1 & \text{otherwise} \end{cases}$.

(ii) ($p = -1$.)

Then $\lfloor\frac{q+r}{p}\rfloor = \begin{cases} -q & \text{if } r = 0 \\ -q-1 & \text{if } r > 0 \end{cases}$ and $\lfloor\frac{-q-r-1}{p}\rfloor = q+1$.

Hence, $A = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r > 0 \end{cases}$.

(iii) ($p < -1$.) In this case, we have

$\lfloor\frac{q+r}{p}\rfloor = \begin{cases} 0 & \text{if } q+r = 0 \\ -1 & \text{if } q+r > 0 \end{cases}$ and $\lfloor\frac{-q-r-1}{p}\rfloor = \begin{cases} 1 & \text{if } q+r+1 \geq -p \\ 0 & \text{if } q+r+1 < -p \end{cases}$.

Hence, $A = \begin{cases} 0 & \text{if } q+r = 0 \text{ or } q+r \geq -p-1 \\ -1 & \text{if } 0 < q+r < -p-1 \end{cases}$.

Therefore the possible values of the expression $A-1$ are -3,-2,-1, and 0.

2.3. More generally, we can prove the following result.

Let $f(x)$ be a nonconstant polynomial with integer coefficients. Then the numbers $f(1)$, $f(2)$, $f(3)$, $\cdots$, contain infinitely many prime factors.

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ and $a_n \neq 0$ for some $n \geq 1$.

(1) If $a_0 = 0$, then $f(k) = k(a_n k^{n-1} + a_{n-1}k^{n-2} + \cdots + a_1)$. As k can be any prime number, the numbers $f(1), f(2), f(3), \cdots$, contain infinitely many prime factors.

(2) Consider the case $a_0 \neq 0$. Suppose the numbers $f(1), f(2), f(3), \cdots$, contain only finitely many prime factors, $p_1, p_2, \ldots, p_m$. Let y be any integer. We have

$$\begin{aligned} f(p_1p_2\cdots p_m a_0 y) &= a_n(p_1p_2\cdots p_m a_0)^n y^n + a_{n-1}(p_1p_2\cdots p_m a_0)^{n-1}y^{n-1} \\ &\quad + \ldots + a_1(p_1p_2\cdots p_m a_0)y + a_0, \\ &= a_0(A_n y^n + A_{n-1}y^{n-1} + \cdots + A_1 y + 1), \end{aligned}$$

where $A_i = a_i(p_1p_2\cdots p_m)^i a_0^{i-1}$, $i = 1, 2, \ldots, n$ and $A_n \neq 0$.

Let $g(y) = A_n y^n + A_{n-1}y^{n-1} + \cdots + A_1 y + 1$. As $p_1p_2\cdots p_m$ divides A_i for all $i = 1, 2, \ldots n$, we have

$$g(y) = \text{multiple of } p_1p_2\cdots p_m \ + \ 1.$$

Therefore, $p_1, p_2, p_3, \ldots, p_m$ are not the factors of $g(y)$.

As the equation $g(y) = \pm 1$ has at most $2n$ roots, we can pick an integer y_0 such that $a_0 y_0 > 0$ and $g(y_0) \neq \pm 1$. Then the integer $g(y_0)$ must have a prime factor p distinct from $p_1, p_2, \ldots, p_m$. Consequently, $f(p_1p_2\cdots p_m a_0 y_0) = a_0 g(y_0)$ has a prime factor different from $p_1, p_2, \ldots, p_m$. This contradicts the assumption that $p_1, p_2, \ldots, p_m$ are all the prime factors of the numbers $f(1), f(2), f(3)\cdots$.

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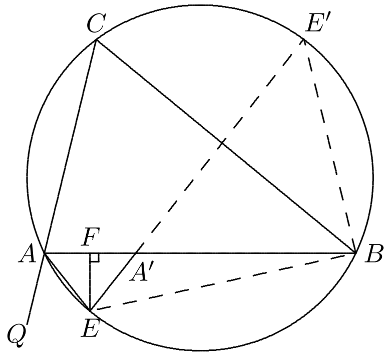


1.1. Let A' be the point on AB such that $A'F = FA$. Then $\triangle AEA'$ is isosceles. Extend EA' meeting the circumcircle of $\triangle ABC$ at E'. Join BE' and BE. Since $\angle ABC = \angle EBC - \angle ABE = \angle AA'E - \angle ABE = \angle E'EB$, we have $BE' = AC$. Also, $\triangle AEA'$ is similar to $\triangle E'BA'$ implies that $A'B = BE' = AC$. Hence, $2AF = AB - A'B = AB - AC$.
(Remark: Let PA be the tangent at A with P inside the sector of $\angle QAE$. As $AB > AC$, we have $\angle C > \angle B$. Hence, $\angle PAB = \angle C > \angle B = \angle QAP$. This implies that E is on the arc AB not containing C. Also, $\angle EBF = \angle PAE < \angle EAB$ so that $BF > AF$. Hence, A' is between F and B.)

1.2. The problem can be changed to find all integers m such that $5^m + 12^m$ is a perfect square. Again the only answer is $m = 2$. We shall give the solution in this more general case. (The solution of the original problem is easy by considering mod 5 or mod 10.)
One solution is $p = 2$ and we assert that it is the only solution. If $p = 2k+1$ is odd, then $5^{2k+1} + 12^{2k+1} \equiv 2^{2k+1} \equiv 2 \cdot 4^k \equiv 2(-1)^k \equiv 2$ or $3 \pmod 5$. However the square of an integer can only be $0, 1$ or $4 \pmod 5$. So $5^p + 12^p$ is not a square when p is odd.
Now suppose that $5^{2n} + 12^{2n} = t^2$ with $n \geq 2$. Then

$$5^{2n} = t^2 - 12^{2n} = (t - 12^n)(t + 12^n).$$

If 5 divides both factors on the right, it must also divide their difference which means it divides 12. But this is impossible. Thus $t - 12^n = 1$ and

$$5^{2n} = 2 \cdot 12^n + 1 \quad \text{or} \quad 2^{2n+1}3^n = (5^n - 1)(5^n + 1).$$

If n is odd, then $3 \mid 5^n + 1$ and $3 \nmid 5^n - 1$. Thus $5^n + 1 = 2 \cdot 3^n$ and $5^n - 1 = 4^n$ which cannot hold for $n > 1$. If n is even, then $5^n - 1 = 2 \cdot 3^n$ and $5^n + 1 = 4^n$, which again cannot hold for $n \geq 2$. Thus there is no solution for $p = 2n$, $n \geq 2$.

1.3. **1st solution by Tan Chee Hau**
We shall prove the assertion using induction on n. Let $x_1, x_2, \ldots, x_n$ be the coordinates of the n red points on the real line. Similarly, let $y_1, y_2, \ldots, y_n$ be the coordinates of the n blue points on the real line. Let A_n be the sum of distances of points of the same colour, B_n the sum of distances of points of different colours. If $n = 1$, then $A_1 = 0$ and $B_1 = |x_1 - y_1|$. Clearly, $B_1 \geq A_1$. Now suppose $B_{n-1} \geq A_{n-1}$.

$$A_n - A_{n-1} = \sum_{i=1}^{n}(x_n - x_i) + (y_n - y_i) = \sum_{i=1}^{n}(x_n - y_i) + (y_n - x_i).$$

$$B_n - B_{n-1} = |x_n - y_n| + \sum_{i=1}^{n} |x_n - y_i| + |y_n - x_i|.$$

Hence, $B_n - B_{n-1} \geq A_n - A_{n-1}$. It follows from this and induction hypothesis that $B_n \geq A_n$.

2nd solution by Lim Yin
Take 2 consecutive points A and B with the coordinate of A less than the coordinate of B. Suppose that there are k blue points and l red points with their coordinates less than or

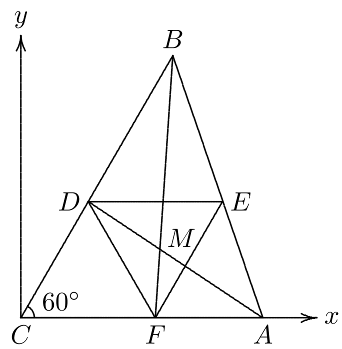
equal to the coordinate of A. Then the segment AB is covered $(n-k)k+(n-l)l$ times by segments whose endpoints have the same colour, and $(n-k)l+(n-l)k$ times by segments whose endpoints have different colours. Since $(n-k)k+(n-l)l \le (n-k)l+(n-l)k$, the assertion follows by summing the lengths of all these segments over all pairs of consecutive points.

3rd solution by Julius Poh
Let S be the total length of the segments whose endpoints are of the same colour and D be the total length of the segments whose endpoints are of different colour. Move the leftmost point to the right by a distance x. Then S decreases by $(n-1)x$ while D decreases by nx. Thus D decreases more than S. Continue to move this point until it hits the next point. If these two points are of different colour, then deleting them causes S and D to decrease by the same amount. If they are of the same colour, then continue to move the pair to the right and in the process D decreases more than S does. We continuing this process, when the block that we are moving (all points in the block are of the same colour) hits a point which is of different colour, remove a pair of points of different colour. If it hits a point of the same colour, then add the point to the block and continue moving to the right. Eventually all the points will be removed and both S and D have decreased to 0. Thus at the beginning $D \ge S$.

2.1. **1st solution**
Let $x=y=1$. We have $f(0)=0$. Let $a=x+y$ and $b=x-y$. Then the given functional equation is equivalent to $bf(a)-af(b)=(a^2-b^2)ab$. This holds for all real numbers a and b. For nonzero a and b, this can be rewritten as

$$\frac{f(a)}{a}-a^2=\frac{f(b)}{b}-b^2.$$

Hence, for any nonzero real number x, $\frac{f(x)}{x}-x^2=f(1)-1$. Let $\alpha=f(1)-1$. We have $f(x)=x^3+\alpha x$, for all $x\ne 0$. As $f(0)=0$, we thus have $f(x)=x^3+\alpha x$ for all $x\in\mathbb{R}$. Clearly $f(x)=x^3+\alpha x$ satisfies the given relation.



2.2. Set up a coordinate system with CA on the x-axis and $C=(0,0)$. Let $A=(a,0)$ with $a>0$, $F=(1,0), D=(\frac{1}{2},\frac{\sqrt{3}}{2})$, and $E=(\frac{3}{2},\frac{\sqrt{3}}{2})$. Then,

$$B=(\frac{a}{2(a-1)},\frac{\sqrt{3}a}{2(a-1)}) \quad \text{and}$$

$$M=(\frac{a(1+a)}{2(1-a+a^2)},\frac{\sqrt{3}a(a-1)}{2(1-a+a^2)}).$$

Hence, $DF=1, DA^2=(\frac{1}{2}-a)^2+\frac{3}{4}=1-a+a^2$, and

$$DM^2=(\frac{a(1+a)}{2(1-a+a^2)}-\frac{1}{2})^2+(\frac{\sqrt{3}a(a-1)}{2(1-a+a^2)}-\frac{\sqrt{3}}{2})^2=\frac{1}{1-a+a^2}.$$

2nd solution by Tay Kah Keng
Since DE is parallel to CA, $\triangle DEB$ is similar to $\triangle FAE$ so that $DB:DE=FE:FA$. As $CDEF$ is a rhombus, we have $DE=FE=DF$. Hence, $DB:DF=FD:FA$. Also, $\angle BDF=\angle DFA=120°$. This shows that $\triangle BDF$ is similar to $\triangle DFA$. Therefore,

$\angle DFB = \angle FAD$. This implies that $\triangle DMF$ is similar to $\triangle DFA$. Consequently, $DF^2 = DM \cdot DA$.

2.3. Let $f(n)$ be the given sum. The summands that appear in $f(n)$ but not in $f(n-1)$ are those of the form $a_p = 1/pn$ where $1 \leq p < n$, $(p, n) = 1$; the summands in $f(n-1)$ but not in $f(n)$ are those of the form $b_p = 1/p(n-p)$ where $1 \leq p < n-p$, $(p, n-p) = 1$, equivalently $(p, n) = 1$. (For example, if $n = 10$, those summands in $f(10)$ but not in $f(9)$ are $\frac{1}{1\times 10}, \frac{1}{3\times 10}, \frac{1}{7\times 10}, \frac{1}{9\times 10}$, while those which are in $f(9)$ but not in $f(10)$ are $\frac{1}{1\times 9}, \frac{1}{3\times 7}$.) Hence summing only over values of p such that $(p, n) = 1$, we have

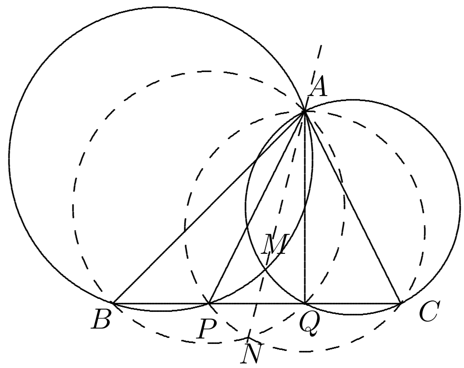
$$f(n) - f(n-1) = \sum_{p<n} a_p - \sum_{2p<n} b_p = \sum_{2p<n} (a_p + a_{n-p} - b_p).$$

But $a_p + a_{n-p} - b_p = 0$; hence $f(n) = f(n-1)$ for all $n \geq 3$, and the result follows.

2000/2001

1.1 It suffices to prove that the product of two differences of two squares is also a difference of two squares. Let $a = x^2 - y^2$ and $b = r^2 - s^2$. Then, $ab = (x-y)(r-s)(x+y)(r+s) = (xr + ys - yr - xs)(xr + ys + yr + xs) = (xr + ys)^2 - (yr + xs)^2$.
There is another characterization of a difference of two squares. Namely, a positive integer n is a difference of two squares of positive integers if and only if $n \neq 1, 2, 4$, and $n \not\equiv 2 \pmod 4$. The result also follows from this characterization.

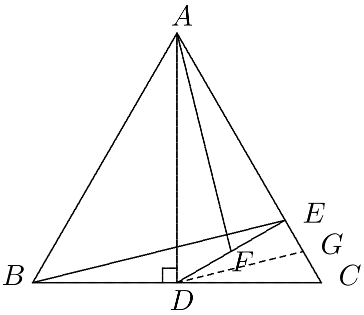
1.2. Let K and L be points of intersection of the line BC with the lines AM and AN respectively. Suppose that the line BC is the x-axis of a coordinate system with origin B, and let c, p, q, k and l denote the x-coordinates of C, P, Q, K and L respectively.



The point K is on the radical axis of the circumcircles of $\triangle PAB$ and $\triangle QAC$, hence its powers $k(k-p)$ and $(k-q)(k-c)$ with respect to these two circles are equal. It follows that $k = cq/(c+q-p)$. Similarly, we have $l = cp/(c+p-q)$, interchanging the roles of p and q. We easily find that $l = k$ if and only if $p + q = c$ and the result follows.

1.3 Each time a player loses a match, he has to wait six games before his turn comes again. If x is the number of games before his first turn, then the player will win if $x + 7r + 7 = 37$, where $r \geq 0$ is an integer and $0 \leq x \leq 6$. Here r counts the number of times he lost. From this, we obtain $x = 2$ and $r = 4$. Thus the second player in the queue wins. That is P_4 wins.

2.1. Let G be the point on CE such that DG is parallel to BE. Then $\angle EBD = \angle GDC$. Also $EG/GC = BD/DC$. Note that $\triangle ADE$ is similar to $\triangle DCE$. Then,



$$\begin{aligned}
& FE/FD = BD/DC \\
\Longleftrightarrow\ & EG/GC = FE/FD \\
\Longleftrightarrow\ & \triangle ADF \text{ is similar to } \triangle DCG \\
\Longleftrightarrow\ & \angle DAF = \angle GDC \\
\Longleftrightarrow\ & \angle DAF = \angle EBD \\
\Longleftrightarrow\ & AF \perp BE.
\end{aligned}$$

2nd Solution Let $A = (0, a), B = (-b, 0), C = (c, 0), D = (0, 0), E = (x, y), F = (tx, ty)$, where $a, , b, c, t > 0$. $DE \perp AC$ implies that $(x, y) = (sa, sc)$ for some s. E lies on AC implies that $s = ac/(a^2 + c^2)$. Hence, $x = a^2c/(a^2 + c^2)$ and $y = ac^2/(a^2 + c^2)$.
Then,

$$\begin{aligned}
& AF \perp BE \\
\Longleftrightarrow\ & (tx, ty - a) \cdot (x + b, y) = 0 \\
\Longleftrightarrow\ & tx^2 + txb + ty^2 - ay = 0 \\
\Longleftrightarrow\ & \tfrac{ta^4c^2}{(a^2+c^2)^2} + \tfrac{tba^2c}{a^2+c^2} + \tfrac{ta^2c^4}{(a^2+c^2)^2} - \tfrac{a^2c^2}{a^2+c^2} = 0 \\
\Longleftrightarrow\ & ta^2c + tb(a^2 + c^2) + tc^3 - c(a^2 + c^2) = 0 \\
\Longleftrightarrow\ & -a^2c(1 - t) - c^3(1 - t) + tb(a^2 + c^2) = 0 \\
\Longleftrightarrow\ & (a^2 + c^2)tb = c(1 - t)(a^2 + c^2) \\
\Longleftrightarrow\ & b/c = (1 - t)/t \\
\Longleftrightarrow\ & BD/DC = FE/FD.
\end{aligned}$$

2.2 For $n \geq 6$, take $x_{n-5} = x_{n-4} = x_{n-3} = x_{n-2} = x_{n-1} = 1/2$ and $x_n = 1$ and zero for other x_i. Then the left hand side of the inequality is $9/4$, while the right hand side is $5/2$. So the inequality is not valid for $n \geq 6$. We shall prove that the inequality holds for $n = 2, 3, 4, 5$. The cases $n = 2$ and 3 can be verified easily. Let's consider the case $n = 5$. (The case $n = 4$ can be proved in a similar way.) The inequality to be proved is equivalent to

$$x_5^2 - (x_1 + x_2 + x_3 + x_4)x_5 + (x_1^2 + x_2^2 + x_3^2 + x_4^2) \geq 0.$$

Regard this as a quadratic equation in x_5. It suffices to prove that its discriminant is less than or equal to zero. The discriminant is equal to $(x_1+x_2+x_3+x_4)^2 - 4(x_1^2+x_2^2+x_3^2+x_4^2)$ which can simplified to $-[(x_1-x_2)^2+(x_1-x_3)^2+(x_1-x_4)^2+(x_2-x_3)^2+(x_2-x_4)^2+(x_3-x_4)^2]$. It is obviously less than or equal to zero.

2.3 If n is a prime-power p^α, where p is a prime and α is a positive integer, then $L(n) = pL(n-1)$ since $p^{\alpha-1} < n$ ensures that $p^{\alpha-1}$ divides $L(n-1)$. On the other hand, if n is not a prime-power, it is greater than every prime-power which divides it, so $L(n) = L(n-1)$. Thus $L(n) = L(n-1)$ if and only if n is not a prime-power.

(i) We shall prove that there are arbitrarily long sequences of consecutive positive integers with the same value of $L(n)$. For any n, let $P(n)$ be the product of all distinct primes $p \leq n$. If $2 \leq r \leq n$, then r divides $L(n)$, so $L(n)P(n) + r$ is a multiple of r. However, it is not a prime-power, for if p^α is a maximal prime-power factor of r, then $p^{\alpha+1}$ is a factor of $L(n)P(n)$, so that $L(n)P(n) + r$ is greater than r but has p^α as a maximal prime-power factor. Therefore, $\{a_n = L(n)P(n) + r \ : \ 1 \leq r \leq n\}$ is a sequence of n consecutive positive integers with the same value of $L(a_n)$. Now take $n = 2001$.

(ii) We know that $m+1, m+2, m+3$ are all prime powers. One of them is a power of 2 and another is a power of 3 and they must be adjacent. Since the equation

$$2^x + 1 = 3^y \tag{1}$$

has two solutions in integers, $(x, y) = (3, 2), (1, 1)$ and the equation

$$2^x - 1 = 3^y \tag{2}$$

has the solution $(x, y) = (2, 1)$. By examining $L(1), \ldots, L(11)$, we see that the only solutions are:

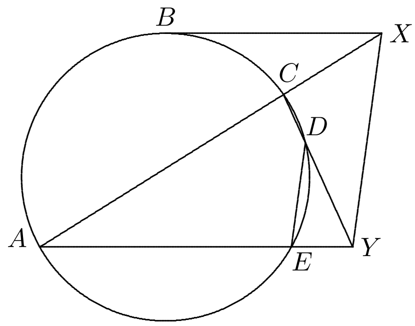
$$m = 1, 2, 6.$$

[(1) has solutions $(x, y) = (3, 2), (1, 1)$. Assume that $x > 3$. As $3^y - 1 = 2^x = 16(2^{x-4})$ is divisible by 16, it implies that $y \equiv 0 \pmod 4$. Write $y = 4k$. Thus $3^{4k} - 1 = 81^k - 1 = 80(81^{k-1} + 81^{k-2} + \cdots + 1)$ which is not a power of 2.

(2) has a solution $(x, y) = (2, 1)$. Now assume that $y > 1$. Then $2^x - 1$ is divisible by 9. This implies that x must be even. Write $x = 2x_1$. Then $2^x = 4^{x_1}$. For $4^{x_1} \equiv 1 \pmod 9$, we have $x_1 = 3k$. Thus $2^x - 1 = 64^k - 1 = 63(64^{k-1} + 64^{k-2} + \cdots + 1)$. Thus it cannot be a power of 3.]

2001/2002

1.1 Suppose $XY = XB$. Then $XY^2 = XB^2 = XC \cdot XA$ so that $XY : XC = XA : XY$. This shows that $\triangle XCY$ is similar to $\triangle XYA$. Hence $\angle EDY = \angle XAY = \angle XYC$. Therefore, XY is parallel to DE. The converse is similar.



1.2 Use Induction. We can prove by induction that $O_n = 2^{2n-1} - 2^{n-1}$ and $E_n = 2^{2n-1} + 2^{n-1}$. We merely have to note that

$$O_{n+1} = E_n + 3O_n, \quad E_{n+1} = 3E_n + O_n.$$

2nd Solution Using generating function:
Let $y_i = x_{2i-1}x_{2i}$. Then $S_n = y_1 + \cdots + y_n$. There are three ways for each y_i to be 0 and 1 way for it to be 1. Thus if $f(x) = (3+x)^n = \sum a_i x_i$, then a_i is the number of sequences with $S_n = i$. Thus $O_n = a_1 + a_3 + \cdots = [f(1) - f(-1)]/2$ and $E_n = a_0 + a_2 + \cdots = [f(1) + f(-1)]/2$. The result thus follows.

3rd Solution Direct computation:
Let $y_i = x_{2i-1}x_{2i}$. Then $S_n = y_1 + \cdots + y_n$. There are three ways for each y_i to be 0 and 1 way for it to be 1. Thus

$$O_n = 3^{n-1}\binom{n}{n-1} + 3^{n-3}\binom{n}{n-3} + 3^{n-5}\binom{n}{n-5} + \cdots$$
$$E_n = 3^n + 3^{n-2}\binom{n}{n-2} + 3^{n-4}\binom{n}{n-4} + \cdots$$

From here we have $E_n + O_n = (1+3)^n$ and $E_n - O_n = (3-1)^n$. The result then follows.

1.3 We show more generally that $ak^2 + bk + c \equiv 0 \pmod{2^n}$ has a solution for all n whenever b is odd and a or c is even. For $n = 1$, take $k = 0$ if c is even and $k = 1$ if c is odd. Now suppose the claim is true for all n. If c is even, then, by assumption, the congruence $2at^2 + bt + c/2 \equiv 0 \pmod{2^n}$ has some solution t. Letting $k = 2t$ we get $ak^2 + bk + c = 2(2at^2 + bt + c/2) \equiv 0 \pmod{2^{n+1}}$. If c is odd, then a is even, so $a + b + c$ is even; hence, by assumption, the congruence $2at^2 + (2a+b)t + (a+b+c)/2 \equiv 0 \pmod{2^n}$ has some solution t. Letting $k = 2t + 1$ yields

$$ak^2 + bk + c = 2[2at^2 + (2a+b)t + (a+b+c)/2] \equiv 0 \pmod{2^{n+1}}.$$

Thus, whether c is even or odd, the claim is true for $n+1$, and so by induction for all n.

2nd Solution by Tan Kiat Chuan, Tay Wei En Joel, Leung Ngai-Hang Zachary, Kenneth Tay
It suffices to show that $2k^2 + 2001k + 3$, $k = 0, 1, \cdots, 2^n - 1$ forms a complete residue class modulo 2^n. Suppose there are distinct integers k_1, k_2, $0 \le k_1, k_2 \le 2^n - 1$ such that $2k_1^2 + 2001k_1 + 3 \equiv 2k_2^2 + 2001k_2 + 3 \pmod{2^n}$. That is $(k_1 - k_2)(2k_1 + 2k_2 - 2001) \equiv$

(mod 2^n). Since $2k_1 + 2k_2 + 2001$ is odd, we must have 2^n divides $k_1 - k_2$. Thus $k_1 \equiv k_2$ (mod 2^n). Since $0 \le k_1, k_2 \le 2^n - 1$, we have $k_1 = k_2$. Therefore, $2k^2 + 2001k + 3$ $k = 0, 1, \cdots, 2^n - 1$ forms a complete residue class modulo 2^n.

2.1 The inequality can be written in the form

$$(\frac{x_1^2 + x_2^2 + x_3^2}{3})^{\frac{1}{2}} \le (\frac{x_1^3 + x_2^3 + x_3^3}{3})^{\frac{1}{3}}$$

This is known as the Power Mean Inequality. There are several proofs of this inequality. First recall Hölder's inequality: Let p, q be real numbers such that $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $2n$ positive real numbers $a_1, b_1, \ldots, a_n, b_n$,

$$\sum_{i=1}^{n} a_i b_i \le (\sum_{i=1}^{n} a_i^p)^{\frac{1}{p}} (\sum_{i=1}^{n} b_i^q)^{\frac{1}{q}}.$$

In our case, take $p = 3$, $q = \frac{3}{2}$, $n = 3$, $a_1 = a_2 = a_3 = 1$ and $b_1 = x_1^2$, $b_2 = x_2^2$, $b_1 = x_3^2$. We then have

$$x_1^2 + x_2^2 + x_3^2 \le 3^{\frac{1}{3}} (x_1^3 + x_2^3 + x_3^3)^{\frac{2}{3}}.$$

That is

$$\frac{(x_1^2 + x_2^2 + x_3^2)^3}{(x_1^3 + x_2^3 + x_3^3)^2} \le 3.$$

Alternatively, Consider the function $f(x) = x^{\frac{3}{2}}$ for $x > 0$. $f''(x) = \frac{3}{\sqrt{x}} > 0$ for $x > 0$. Hence, f is concave upward. By Jensen's Inequality, for any three positive numbers z_1, z_2, z_3,

$$f\left(\frac{z_1 + z_2 + z_3}{3}\right) \le \frac{f(z_1) + f(z_2) + f(z_3)}{3}.$$

Now take $z_1 = x_1^2$, $z_2 = x_2^2$ and $z_3 = x_3^2$. We have

$$\left(\frac{x_1^2 + x_2^2 + x_3^2}{3}\right)^{\frac{3}{2}} \le \frac{x_1^3 + x_2^3 + x_3^3}{3}.$$

That is

$$\frac{(x_1^2 + x_2^2 + x_3^2)^3}{(x_1^3 + x_2^3 + x_3^3)^2} \le 3.$$

2nd Solution by Lim Yin

The given inequality is equivalent to

$$\begin{aligned} &[x_1^6 + 2x_1^3x_2^3 - 3x_1^4x_2^2] + [x_1^6 + 2x_1^3x_3^3 - 3x_1^4x_3^2] + [x_2^6 + 2x_2^3x_1^3 - 3x_2^4x_1^2] \\ +&[x_2^6 + 2x_2^3x_3^3 - 3x_2^4x_3^2] + [x_3^6 + 2x_3^3x_1^3 - 3x_3^4x_1^2] + [x_3^6 + 2x_3^3x_2^3 - 3x_3^4x_2^2] \\ +&[2x_1^3x_2^3 + 2x_1^3x_3^3 + 2x_2^3x_3^3 - 6x_1^2x_2^2x_3^2] \ge 0. \end{aligned}$$

Each term in the square brackets is non-negative by the AM-GM inequality. So the result follows.

3rd Solution by Leung Ngai-Hang Zachary

The given inequality is equivalent to:

$$\begin{aligned} &[2x_1^3x_2^3 + 2x_1^3x_3^3 + 2x_2^3x_3^3 - 6x_1^2x_2^2x_3^2] + [x_1^6 + x_2^6 + 4x_1^3x_2^3 - 3x_1^4x_2^2 - 3x_1^2x_2^4] \\ +&[x_1^6 + x_3^6 + 4x_1^3x_3^3 - 3x_1^4x_3^2 - 3x_1^2x_3^4] + [x_2^6 + x_3^6 + 4x_2^3x_3^3 - 3x_2^4x_3^2 - 3x_2^2x_3^4] \ \ge \ 0. \end{aligned}$$

The first term is nonnegative by rearrangement inequality. The next three can be shown to be nonnegative by using rearrangement inequality as follows:

$$\begin{aligned} x_1^6+x_2^6+4x_1^3x_2^3 &= x_1^3(x_1^3+x_2^3)+x_2^3(x_2^3+x_1^3)+2x_1^3x_2^3 \\ &\geq x_1^3(x_1^2x_2+x_1x_2^2)+x_2^3(x_1x_2^2+x_1^2x_2)+2x_1^3x_2^3 \\ &= x_1^5x_2+x_1^4x_2^2+x_1x_2^5+x_1^2x_2^4+2x_1^3x_2^3 \\ &= (x_1^5x_2+x_1^3x_2^3)+(x_1x_2^5+x_1^3x_2^3)+x_1^4x_2^2+x_1^2x_2^4 \\ &\geq 3x_1^4x_2^2+3x_1^2x_2^4. \end{aligned}$$

Alternatively,

$$x_1^6+x_2^6+4x_1^3x_2^3 = (x_1^6+x_1^3x_2^3+x_1^3x_2^3)+(x_2^6+x_1^3x_2^3+x_1^3x_2^3) \geq 3x_1^4x_2^2+3x_1^2x_2^4.$$

2.2 Suppose r is not an integer, choose an integer a such that $ar \neq \lfloor ar \rfloor > 1$. (Note that $r > 0$. If r is irrational, choose any large positive integer a. If $r = p/q$ is rational, choose a large positive integer a such that $(a, q) = 1$.) Let k be the unique integer such that

$$\frac{1}{k+1} \leq ar - \lfloor ar \rfloor < \frac{1}{k}.$$

Then

$$1 \leq (k+1)(ar - \lfloor ar \rfloor) < \frac{k+1}{k} \leq 2.$$

Since

$$\lfloor (k+1)ar \rfloor = (k+1)\lfloor ar \rfloor + \lfloor (k+1)(ar - \lfloor ar \rfloor) \rfloor = (k+1)\lfloor ar \rfloor + 1$$

we see that $\lfloor ar \rfloor$ does not divide $\lfloor (k+1)ar \rfloor$. Thus $m = a$, $n = (k+1)a$ form a counter example.

2nd Solution by Charmaine Sia

Suppose that r is not an integer. We may assume $r > 1$. (If $0 < r < 1$, we may choose a positive integer p such that $pr > 1$ and pr is not an integer. Then consider $r' = pr$.) Now choose a positive integer m such that $k + \frac{1}{2} \leq mr < k + 1$ for some positive integer $k > 1$. Then $\lfloor mr \rfloor = k$ and $\lfloor 2mr \rfloor = 2k + 1$. Thus $\lfloor mr \rfloor$ does not divide $\lfloor 2mr \rfloor$.

3rd Solution by Tan Weiyu Colin

Given r, choose $m \in \mathbb{N}$ so that $mr > 10$ and $mr \notin \mathbb{N}$. Let $mr = a + \frac{b}{10^j} + \frac{x}{10^{j+1}}$ where $j \in \mathbb{N}$, $a = \lfloor mr \rfloor > 10$, b is a positive integer less than 10 and x is a nonnegative real number, also less than 10. Let $k = 10^j$ and $n = mk$. Then $\lfloor mr \rfloor = a$, $\lfloor nr \rfloor = ka + b$ and $\lfloor mr \rfloor = a \nmid \lfloor nr \rfloor$.

2.3 Fix any $x \geq 0$. Let $f^{[0]}(x) = x$ and $f^{[1]}(x) = f(x)$. For $n \geq 1$, let $f^{[n]}(x) = f(f^{[n-1]}(x))$. Then the above functional equation gives

$$f^{[n+2]}(x) + f^{[n+1]}(x) = 12 f^{[n]}(x).$$

Solving this difference equation, we have

$$f^{[n]}(x) = C_1 3^n + C_2(-4)^n.$$

Using the initial conditions $f^{[0]}(x) = x$ and $f^{[1]}(x) = f(x)$, we have $C_1 = (f(x) + 4x)/7$ and $C_2 = (3x - f(x))/7$. Therefore,

$$f^{[n]}(x) = \frac{1}{7}(f(x) + 4x)3^n + \frac{1}{7}(3x - f(x))(-4)^n.$$

Since $f(x) \geq 0$, $f^{[n]}(x) \geq 0$ for all $n \geq 0$. By taking n even, we have $\frac{1}{7}(f(x)+4x)3^n + \frac{1}{7}(3x - f(x))4^n \geq 0$. From this, $3x - f(x) \geq 0$. By taking n odd, we have $\frac{1}{7}(f(x)+4x)3^n - \frac{1}{7}(3x - f(x))4^n \geq 0$. From this, $3x - f(x) \leq 0$. Consequently, $f(x) = 3x$. One can easily verify that $f(x) = 3x$ satisfies the given functional equation.
Alternatively, suppose for some a, $f(a) = 3a + c$ where $c \in \mathbb{R}$. Then $f^{[n]}(a) = 3^n a + k_n c$. One can obtain a recurrence relation in k_n and use it to prove that $c = 0$. Finally check that $f(x) = 3x$ satisfies the given condition.

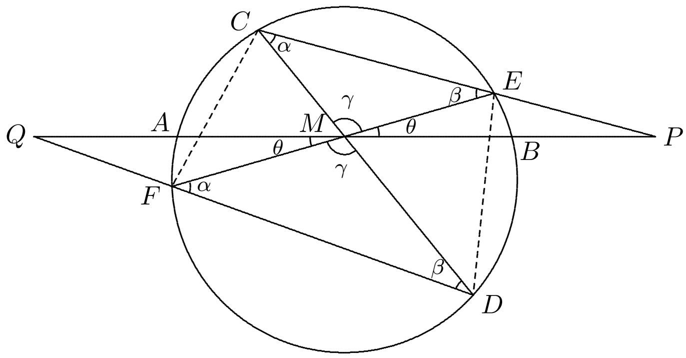
2002/2003

1.1 Take $n = 10^{222} - 3$. Then

$$(10^{222} - 3)^2 = 10^{444} - 6 \cdot 10^{222} + 9 = \underbrace{9 \cdots 9}_{221 \text{ nines}} 40 \cdots 09.$$

The sum of the digits of $n^2 = 222 \times 9 + 4 = 2002$.

1.2 This is in fact the Butterfly Theorem. There are many proofs of this result. See the discussion on page 45 in Geometry Revisited by Coxeter and Greitzer. Here we give two proofs.



First, apply sine rule to $\triangle CMP$,

$$\frac{MP}{\sin\alpha} = \frac{CP}{\sin(\gamma+\theta)} \quad - \quad (1)$$

Apply sine rule to $\triangle EMP$,

$$\frac{MP}{\sin\beta} = \frac{EP}{\sin\theta} \quad - \quad (2)$$

(1) $\times$ (2) gives $\quad \dfrac{MP^2}{\sin\alpha\sin\beta} = \dfrac{CP \cdot EP}{\sin(\gamma+\theta)\sin\theta}. \quad - \quad (3)$

For $\triangle QDM$, $\quad \dfrac{QM}{\sin\beta} = \dfrac{QD}{\sin(\gamma+\theta)}. \quad - \quad (4)$

For $\triangle QFM$, $\quad \dfrac{QM}{\sin\alpha} = \dfrac{QF}{\sin\theta}. \quad - \quad (5)$

(4) $\times$ (5) gives $\quad \dfrac{QM^2}{\sin\alpha\sin\beta} = \dfrac{QD \cdot QF}{\sin(\gamma+\theta)\sin\theta}. \quad - \quad (6)$

By (3) and (6), $\quad \dfrac{CP \cdot EP}{MP^2} = \dfrac{QD \cdot QF}{QM^2}$

This implies $\quad \dfrac{(MP+AM)(MP-AM)}{MP^2} = \dfrac{(QM+AM)(QM-AM)}{QM^2}.$

Or equivalently, $\quad \dfrac{MP^2 - AM^2}{MP^2} = \dfrac{QM^2 - AM^2}{QM^2}.$

That is $\quad MP = QM.$

Second solution Set up a rectangular coordinate system with M as the origin and QP as the x-axis. Let the equation of the circle be $x^2+(y+c)^2=r^2$. Let the coordinates of C, D, F and E be (p_1, ap_1) and (p_2, ap_2) (q_1, bq_1), (q_2, bq_2), respectively. Direct calculation shows that the x-intercept of CE at the point P is $p_1q_2(b-a)/(bq_2-ap_1)$ and the x-intercept of DF at Q is $p_2q_1(a-b)/(ap_2-bq_1)$. It suffices to verify that $p_1q_2/(bq_2-ap_1) = p_2q_1/(ap_2-bq_1)$. This equation can be rearranged to

$$ap_1p_2(q_1+q_2) = bq_1q_2(p_1+p_2). \qquad (*)$$

The line CD has equation $y = ax$. Thus the x-coordinates of C and D are the roots of the equation $x^2 + (ax+c)^2 = r^2$. That is p_1 and p_2 are the roots of the quadratic equation $(1+a^2)x^2+2acx+(c^2-r^2) = 0$. Similarly, q_1 and q_2 are the roots of the quadratic equation $(1+b^2)x^2+2bcx+(c^2-r^2) = 0$. Using the relations between roots and coefficients, we have $p_1+p_2 = -2ac/(1+a^2)$ and $p_1p_2 = (c^2-r^2)/(1+a^2)$. Similarly, $q_1+q_2 = -2bc/(1+b^2)$ and $q_1q_2 = (c^2-r^2)/(1+b^2)$. With these, $(*)$ can be easily verified.

1.3 If $\max_j \min_i a_{ij} = \min_i \max_j a_{ij} = a_{\alpha\beta}$, then clearly $a_{\alpha\beta}$ is at once the largest number in the αth row and the smallest numbers in the βth column, and hence

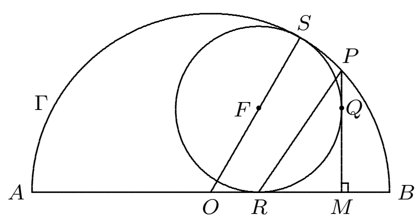
$$a_{\alpha j} < a_{\alpha\beta} < a_{i\beta} \text{ for all } i \neq \alpha \text{ and for all } j \neq \beta. \qquad (*)$$

Conversely, if $(*)$ holds for some $a_{\alpha\beta}$, then $\min_i a_{ij} \le a_{\alpha j} < a_{\alpha\beta}$ for all $j \neq \beta$ and $\max_j a_{ij} \ge a_{i\beta} > a_{\alpha\beta}$ for all $i \neq \alpha$ would imply that $\max_j \min_i a_{ij} = a_{\alpha\beta} = \min_i \max_j a_{ij}$. To obtain a required configuration, it is therefore necessary and sufficient to choose any $2n-1$ of the given n^2 numbers, say $x_1 < x_2 < \cdots < x_{2n-1}$. Put x_n anywhere in the array. Then put $x_1, x_2, \ldots, x_{n-1}$ in the same row as x_n and put $x_{n+1}, x_{n+2}, \ldots, x_{2n-1}$ in the same column as x_n. The remaining n^2-2n+1 numbers can be used to fill up the remaining n^2-2n+1 positions. Therefore, the total numbers of such configurations is

$$\binom{n^2}{2n-1} \cdot n^2 \cdot [(n-1)!]^2 \cdot (n^2-2n+1)! = \frac{(n^2)!(n!)^2}{(2n-1)!}.$$

2.1 The answer is 290. First observe that the arithmetic sequences $\{p+mk, k \in \mathbb{N}\}$ and $\{q+nk, k \in \mathbb{N}\}$ are disjoint if and only if $p-q \neq ln-km$ for all integers k, l, which holds if and only if $\gcd(m,n)$ does not divide $p-q$. Therefore, the required r cannot be relatively prime to $10 = (2)(5)$, $26 = (2)(13)$ and 29. We start by choosing r to be the smallest lcm of d_1, d_2, d_3 where d_1, d_2, d_3 are factors (greater than 1) of 10, 26 and 29 respectively. The smallest such r is $58 = (2)(29)$. In this case, $\gcd(10,58) = 2$, $\gcd(26,58) = 2$ and $\gcd(29,58) = 29$. We also require $b-3 \not\equiv 0 \pmod 2$, $b-6 \not\equiv 0 \pmod 2$ and $b-5 \not\equiv 0 \pmod{29}$. But there is no solution for b from the first two equations. Therefore we cannot take $r = 58$. The next smallest lcm would be $290 = (5)(2)(29)$. In this case, a simple checking using the above criterion shows that $\{1+290k, k \in \mathbb{N}\}$ is disjoint from A

2.2 Let the radius of Γ be r_1 and the radius of the inscribed circle be r_2 and its center be F. First we have $PB^2 = AB \cdot BM$ because $\triangle APB$ is similar to $\triangle PMB$.
Using Pythagoras' Theorem, $RO^2 = FO^2 - FR^2 = (r_1-r_2)^2 - r_2^2 = r_1^2 - 2r_1r_2 = AO^2 - 2AO \cdot RM$.



Thus, $AO^2 - RO^2 = 2AO \cdot RM = AB \cdot RM$. Therefore, $AB \cdot RB - RB^2 = AR \cdot RB = AO^2 - RO^2 = AB \cdot RM$. From this, we have $AB \cdot (RB - RM) = RB^2$. Therefore, $PB^2 = AB \cdot BM = AB \cdot (RB - RM) = RB^2$ and so $PB = RB$.

Note that since $\angle PAM = \angle BPM$, we have $PB = RB$ if and only if $\angle BPR = \angle BRP$ if and only if $\angle APR = \angle MPR$ if and only if PR bisects $\angle APM$. With this observation, if we inscribe another circle in the curvilinear triangle PBM touching MB at a point R', then $\angle RPR' = 45°$. Note also that S, Q, B are in fact collinear. Using this, we can obtain another solution as follow. Let the extension of PM meet the circle at M'. Then $BR^2 = BQ \cdot BS = BQ^2 + BQ \cdot QS = BQ^2 + PQ \cdot QM' = BQ^2 + (PM - MQ)(M'M + MQ) = BQ^2 + (PM - MQ)(PM + MQ) = BQ^2 + PM^2 - MQ^2 = MB^2 + PM^2 = PB^2$. There is an even shorter proof due to Colin Tan. The fact that S, Q, B are collinear gives $\angle ASQ = 90°$. Thus $\triangle ABS$ is similar to $\triangle QBM$ so that $AB/BS = BQ/BM$. Therefore, $PB^2 = AB \cdot BM = BS \cdot BQ = BR^2$.

2.3 Replacing m by $f(f(m))$, we have

$$f(f(f(m)) + f(f(n))) = -f(f(f(f(m)) + 1)) - n. \qquad (1)$$

Interchanging m and n in (1), we get

$$f(f(f(m)) + f(f(n))) = -f(f(f(f(n)) + 1)) - m. \qquad (2)$$

Put $m = 1$ in the original functional equation and denote for simplicity $f(f(2))$ by k. We obtain $f(f(f(n)) + 1) = -k - n$. Using this and equating (1) and (2), we get $f(-m-k) - f(-n-k) = m - n$. Letting $m = -p + k$ and $n = -p + k + 1$, we have $f(p) - f(p-1) = -1$. Inductively, we obtain $f(p) = f(0) - p$. Thus, $f(f(p)) = f(f(0) - p) = f(0) - (f(0) - p) = p$. Substituting this into the original functional equation, we have $f(m+n) = -m - 1 - n$. In other words, $f(n) = -n - 1$ for all n. Indeed, $f(n) = -n - 1$ satisfies the given functional equation.

(2nd solution by Teo Wei Hao) Setting $m = 0$, we obtain $f^3(n) = -f^2(1) - n$. This functional relation immediately shows that f is bijective, because $f\left(f^2(n)\right) = -f^2(1) - n$ and $f(p) = f(q) \implies f^3(p) = f^3(q) \implies p = q$. Now we may let $f^2(n_0) = 1$ for some n_0. The original functional equation becomes $f(m+1) = -f^2(m+1) - n_0$. Using the fact that f is bijective, we may let $f(m+1) = x$, so that $f(x) = -x - n_0$. Applying this on $f^2(n_0) = 1$ gives $n_0 = 1$. Therefore, $f(x) = -x - 1$.

(3rd solution by Colin Tan) Let $f^2(1) = c$. Put $n = 1$, $m = 0$ and replace n by $f(n)$, we get, respectively,

$$\begin{aligned} f(m+c) &= -f^2(m+1) - 1 && (1) \\ f^3(n) &= -c - n && (2) \\ f(m + f^3(n)) &= -f^2(m+1) - f(n) && (3) \end{aligned}$$

From (1) and (3), we get

$$f(m+c) + 1 = f(m - c - n) + f(n) \qquad (4)$$

Put $n = -2c + 1$ and replace m by $m + 1 - c$ in (4), we get

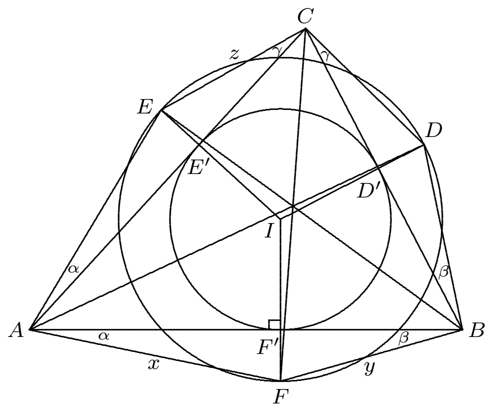
$$f(m+1) + 1 = f(m) + f(-2c+1).$$

Thus f is linear. Write $f(x) = ax + b$ and from the original equation, we get $a = b = -1$. Thus $f(x) = -x - 1$. From the above computation, we see that this function satisfies the original functional equation.

2003/2004

1.1 Let $M = x(x+1)(x+2)(x+3)(x+4)(x+5)(x+6)(x+7)$, where x is a positive integer. Then $M = (x^2 + 7x)(x^2 + 7x + 6)(x^2 + 7x + 10)(x^2 + 7x + 12)$. Let $a = x^2 + 7x + 6 = (x+1)(x+6) \geq 2 \times 7 = 14$. Thus $M = (a-6)a(a+4)(a+6) = a^4 + 4a(a+3)(a-12) > a^4$. Also $(a+1)^4 - M = 42a^2 + 148a + 1 > 0$. Therefore, $a^4 < M < (a+1)^4$. Consequently, $\lfloor N \rfloor = \lfloor M^{\frac{1}{4}} \rfloor = a = (x+1)(x+6)$ which is an even integer.

1.2 Let the intersection of AD, BE, CF with BC, CA, AB be D', E', F' respectively. It is easy to establish that $\angle FAF' = \angle EAE' = \alpha$, $FBF' = \angle DBD' = \beta$, $\angle DCD' = \angle ECE' = \gamma$. Also $AE = AF = x$, $BF = BD = y$, $CD = CE = z$. The ratio $AF'/F'B$ equals to the ratio of the altitudes from A and B on CF of the triangles AFC and BFC and hence as the ratio of their areas.



Therefore, $\dfrac{AF'}{F'B} = \dfrac{\text{Area}\triangle AFC}{\text{Area}\triangle BFC} = \dfrac{xAC\sin(\angle A + \alpha)}{yBC\sin(\angle B + \beta)}$.

Similarly, $\dfrac{BD'}{D'C} = \dfrac{yAB\sin(\angle B + \beta)}{zAC\sin(\angle C + \gamma)}$ and $\dfrac{CE'}{E'A} = \dfrac{zBC\sin(\angle C + \gamma)}{xAB\sin(\angle A + \alpha)}$.

It follows that $\dfrac{AF'}{F'B}\dfrac{BD'}{D'C}\dfrac{CE'}{E'A} = 1$, so by Ceva's Theorem, AD, BE and CF are concurrent.

1.3 The integer pair (x, y) is a solution of the given equation if and only if $x + y = 0$ or $(x, y) = (0 \pm 1)$, $(\pm 1, 0)$, or $\pm(2, 2)$. Clearly, if $x + y = 0$, then (x, y) is a solution. Assume now that (x, y) is a solution with $x + y \neq 0$.

We first show that $xy \geq 0$. Dividing both sides of $x^5 + y^5 = (x + y)^3$ by $x + y$ yields

$$x^4 - x^3y + x^2y^2 - xy^3 + y^4 = (x + y)^2.$$

This is equivalent to

$$(x^2 + y^2)^2 + x^2y^2 = (x + y)^2(xy + 1),$$

and it follows that $xy \geq 0$.

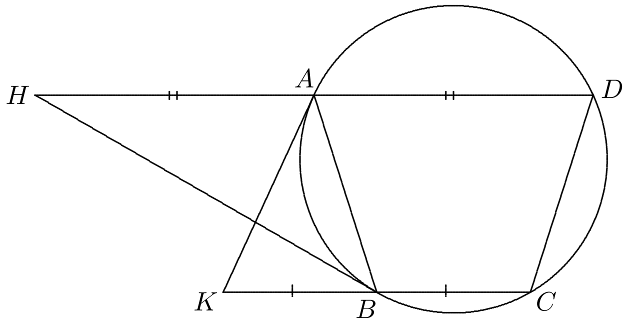
Next we show that $|x + y| \leq 4$. The convexity of the function $f(t) = t^5$ on $[0, \infty)$ implies that for nonnegative x and y,

$$\frac{x^5 + y^5}{2} \geq \left(\frac{x + y}{2}\right)^5, \text{ or equivalently, } x^5 + y^5 \geq \frac{1}{16}(x + y)^5.$$

If $x+y>4$, then $x^5+y^5>(x+y)^3$. Similarly, if x and y are both nonpositive with $x+y<-4$, then $x^5+y^5<(x+y)^3$.

Finally, examining the cases where $xy \geq 0$ and $|x+y| = 1, 2, 3$ or 4, we find the solutions $(x, y) = (0 \pm 1)$, $(\pm 1, 0)$, or $\pm(2, 2)$.

2.1 We have to show that either AB is parallel to CD or AD is parallel to BC. Using the powers of H nd K respective to the circle, we have $HB^2 = HA \cdot HD = 2HA^2$ and $KA^2 = KB \cdot KC = 2KB^2$. Thus $HB/HA = KA/KB$.



Note that $\angle HBA = \angle KAB$. Applying sine rule to triangles ABK and BAH, we obtain $KA/KB = \sin(\angle ABK)/\sin(\angle KAB)$ and $HB/HA = \sin(\angle HAB)/\sin(\angle HBA)$. Thus $\sin(\angle ABK) = \sin(\angle HAB)$. Therefore, either $\angle ABK = \angle HAB$ or $\angle ABK + \angle HAB = 180^\circ$. Consequently, AB is parallel to CD or AD is parallel to BC.

2.2 The smallest such value of k is $1/4$. First note that for $x, y > 0$,

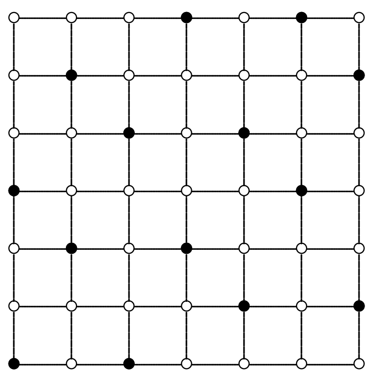
$$\frac{1}{x+y} = \frac{4xy}{x+y} \cdot \frac{1}{4xy} \leq \frac{(x+y)^2}{(x+y)} \cdot \frac{1}{4xy} = \frac{1}{4}\left(\frac{1}{x} + \frac{1}{y}\right),$$

with equality if and only if $x = y$. We then have $\dfrac{ab}{a+b+2c} + \dfrac{bc}{b+c+2a} + \dfrac{ca}{c+a+2b}$

$$\leq \frac{ab}{4}\left(\frac{1}{a+c} + \frac{1}{b+c}\right) + \frac{bc}{4}\left(\frac{1}{c+a} + \frac{1}{b+a}\right) + \frac{ca}{4}\left(\frac{1}{c+b} + \frac{1}{a+b}\right) = \frac{1}{4}(a+b+c),$$

with equality if and only if $a+b = b+c = c+a$, that is $a = b = c$.

2.3 Consider an $n \times n$ square lattice with $M(n)$ black points so that every square path has at least one black point on it.



Let P be a black point in the lattice, and suppose S is a 2×2 square path that passes through P. Assign P a "credit" of $\frac{1}{k}$ if S passes through exactly k black points. Let $T(P)$ be the sum of all credits assigned to P as S varies over all 2×2 square paths that pass through P.

Note that the sum of $T(P)$ as P varies over all black points in the square lattice is $(n-1)^2$ since each of the $(n-1)^2$ two by two square path contributes 1 to the total.

It is clear that $T(P) \leq 1$ if P is at a corner, and $T(P) \leq 2$ if P is on an outer edge. Suppose P is a point in the interior of the lattice. It lies on exactly four 2×2 square paths, and there must be at least one black point on the 3×3 square path surrounding P. Thus, for such a P, $T(P) \leq 7/2$. Therefore, in all cases, $\frac{7}{2}M(n) \geq (n-1)^2$, or equivalently, $\frac{2}{7}(n-1)^2 \leq M(n)$.

On the other hand, the pattern shown in the figure for a 7×7 lattice ($2/7$ of the points are black and every square path passes through a black point) can be extended to an arbitrary $n \times n$ lattice by tiling an $m \times m$ lattices, $m = 7\lceil n/7 \rceil$, with copies of the lattice in the figure, and then removing $(m - n)$ rows and columns from the top and left respectively, so that the number of black point is less than or equal to $\frac{2}{7}n^2$.