Singapore International Mathematical Olympiad 2009
Senior Team Training

Quiz

1. Let $p$ and $q$ be distinct odd primes. Prove that

$$\sum_{\substack{0 < j < \frac{p}{2} \atop j \text{ odd}}} \left\lfloor \frac{qj}{p} \right\rfloor \equiv \sum_{\substack{\frac{p}{2} < i < p \atop i \text{ even}}} \left\lfloor \frac{qi}{p} \right\rfloor \pmod{2}.$$ 

In the graph of $y = qx/p$, Let $A = (0, 0)$, $B = (p, 0)$, $C = (p, q)$, $D = (0, q)$, $X = (p/2, 0)$, $Y = (p/2, q/2)$ and $Z = (p/2, q)$. There are $p - 1$, which is even, lattice points on each vertical line $x = k$, in the interior of rectangle $ABCD$. If $a_k$ is the number of lattice points that are below the line $AC$ $b_k$ is the number of lattice points above the line $AC$. Then $a_k + b_k = p - 1$. Thus $a_k \equiv b_k \pmod{2}$.

Let $\alpha$ be the number of lattice points with even $x$-coordinate in the region $XBCY$, $\beta$ be the number of lattice points with even $x$-coordinates in the region $CYZ$ and $\gamma$ be the number of lattice points with odd $x$-coordinates in the region $AXY$. Then $\alpha$ is the lhs and $\gamma$ is the rhs. From the above consideration $\alpha \equiv \beta \pmod{2}$. Also note that the number of lattice points in the region $CYZ$ with $x$-coordinate $\lceil p/2 \rceil + i$ is equal to the number of lattice points in the region $AXY$ with $x$-coordinate $\lfloor p/2 \rfloor - i$. Moreover, $\lceil p/2 \rceil$ and $\lfloor p/2 \rfloor$ have opposite parity. Thus $\beta \equiv \gamma \pmod{2}$. 

1
2. Let $M$ be a point on the plane containing a triangle $ABC$. The lines $MA, MB$ and $MC$ intersect the lines $BC, CA$ and $AB$ at $D, E$ and $F$ respectively. The circumcircle of $\triangle DEF$ meets the lines $BC, CA$ and $AB$ respectively at $D', E'$ and $F'$. Prove that $AD', BE'$ and $CF'$ are concurrent.

Using Ceva’s Theorem, we have $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$. Also, $BD \cdot BD' = BF \cdot BF'$, $CE \cdot CE' = CD \cdot CD'$ and $AF \cdot AF' = AE \cdot AE'$. Thus

$$\frac{BD'}{DC'} \cdot \frac{CE'}{EA'} = \frac{BF}{BD} \cdot \frac{CD}{CE} \cdot \frac{AE}{AF} = 1.$$ 

Thus $\frac{BD'}{DC} \cdot \frac{CE'}{EA} \cdot \frac{AF'}{FB} = 1$. By the converse of Ceva’s Theorem, $AD', BE'$ and $CF'$ are concurrent.
3. Let $a, b, c, d$ be nonnegative real numbers. Show that
\[
\left( \frac{abc + bcd + cda + dab}{4} \right)^2 \leq \left( \frac{ab + bc + cd + da + ac + bd}{6} \right)^3.
\]

[Remark: The proof I have is unsatisfying in many respects. You are welcome to contribute better proofs.]

First observe that the inequality to be shown is symmetric in the variables $a, b, c, d$ and homogeneous. If $d = 0$, the inequality follows easily from AM-GM: \((abc)^{2/3} \leq \frac{ab + ac + bc}{3}\). Otherwise, we may assume that $a \geq b \geq c \geq d = 1$. Denote by $C$ the number $(abc)^{1/3}$. By our assumption, $C \geq 1$.

**Claim.** The function
\[
f(x) = \left( \frac{x + C}{2} \right)^3 - \left( \frac{3x + C^3}{4} \right)^2
\]
is increasing for $x \geq C^2$.

The only proof I have of the Claim is to use differential calculus to show that $f'(x) \geq 0$ for $x \geq C^2$.

Assuming the claim, we find that for $x \geq C^2$,
\[
f(x) \geq f(C^2) = \frac{C^3}{16}(C^3 - 3C + 2) = \frac{C^3}{16}(C - 1)(C^2 + C - 2) \geq 0
\]
since $C \geq 1$. By AM-GM inequality $\frac{ab + ac + bc}{3} \geq C^2$. Thus $f(\frac{ab + ac + bc}{3}) \geq 0$. Thus
\[
\left( \frac{ab + ac + bc + C^3}{4} \right)^2 \leq \left( \frac{ab + ac + bc + 3C}{6} \right)^3.
\]
By AM-GM inequality, $3C \leq a + b + c$. Hence
\[
\left( \frac{ab + ac + bc + abc}{4} \right)^2 \leq \left( \frac{ab + ac + bc + a + b + c}{6} \right)^3,
\]
which is the inequality required (setting $d = 1$).

4. Let $ABCDEF$ be a convex hexagon inscribed in a circle. Assume that $AB = BC$, $CD = DE$ and $EF = FA$. Show that
\[
\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.
\]
Consider the cyclic quadrilateral $ACEF$. By Ptolemy’s Theorem,
\[
AC \cdot EF + CE \cdot FA = AE \cdot FC.
\]
Dividing by $FC(AC + CE)$ and using the fact that $EF = FA$, we have
\[
\frac{FA}{FC} = \frac{AE}{AC + CE}.
\]
Similarly,
\[
\frac{BC}{BE} = \frac{AC}{CE + AE},
\frac{DE}{DA} = \frac{CE}{AE + AC}.
\]
Let $AC = a$, $CE = b$ and $AE = c$. Then
\[
\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} = \frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b}.
\]
By AM-HM inequality,
\[
\frac{3}{\frac{1}{b+c} + \frac{1}{a+c}} \leq \frac{b + c + a + b + a + c}{3}.
\]
Hence
\[
\frac{9}{2} \leq (a + b + c)\left(\frac{1}{b+c} + \frac{1}{a+b} + \frac{1}{a+c}\right) = \frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b} + 3.
\]
The desired inequality follows.

5. Let $k$ be a given positive integer. Find the smallest $n$ in terms of $k$ so that for any set $A$ of $n$ integers, there are always two elements in $A$ whose sum or difference is divisible by $2k$.

Consider a set of $n$ integers and list their values mod $2k$ as $r_1, r_2, \ldots, r_n$. In particular, $0 \leq r_i \leq 2k - 1$ for each $i$. The numbers $\{0, \ldots, 2k - 1\}$ can be formed into $k + 1$ groups
\[
\{0\}, \{1, 2k - 1\}, \ldots, \{k - 1, k + 1\}, \{k\}.
\]
If $n \geq k + 2$, there are $i, j$, $i \neq j$, so that $r_i$ and $r_j$ are in the same group. Then either $r_i + r_j$ or $r_i - r_j = 0 \mod 2k$.

On the other hand, if $n \leq k + 1$, we consider the set $\{0, 1, \ldots, k\}$. For any two elements, neither the sum nor the difference is divisible by $2k$.

Thus the smallest $n$ satisfying the condition of the problem is $k + 2$. 