Singapore International Mathematical Olympiad 2009
Senior Team Training

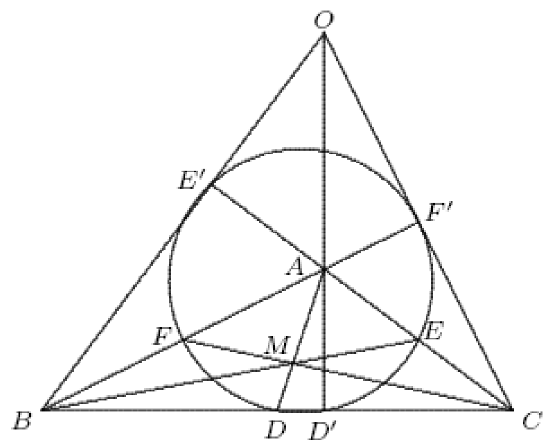
Quiz

1. Let p and q be distinct odd primes. Prove that

$$\sum_{\substack{0<j<p/2 \\ j \text{ odd}}} \left\lfloor \frac{qj}{p} \right\rfloor \equiv \sum_{\substack{p/2<i<p \\ i \text{ even}}} \left\lfloor \frac{qi}{p} \right\rfloor \pmod 2.$$

In the graph of $y = qx/p$, Let $A = (0,0)$, $B = (p,0)$, $C = (p,q)$, $D = (0,q)$, $X = (p/2,0)$, $Y = (p/2,q/2)$ and $Z = (p/2,q)$. There are $p-1$, which is even, lattice points on each vertical line $x = k$, in the interior of rectangle $ABCD$. If a_k is the number of lattice points that are below the line AC b_k is the the number of lattice points above the line AC. Then $a_k + b_k = p-1$. Thus $a_k \equiv b_k \pmod 2$.
Let α be the number of lattice points with even x-coordinate in the region $XBCY$, β be the number of lattice points with even x-coordinates in the region CYZ and γ be the number of lattice points with odd x-coordinates in the region AXY. Then α is the lhs and γ is the rhs. From the above consideration $\alpha \equiv \beta \pmod 2$. Also note that the number of lattice points in the region CYZ with x-coordinate $\lceil p/2 \rceil + i$ is equal to the number of lattice points in the region AXY with x-coordinate $\lfloor p/2 \rfloor - i$. Moreover, $\lceil p/2 \rceil$ and $\lfloor p/2 \rfloor$ have opposite parity. Thus $\beta \equiv \gamma \pmod 2$.

2. Let M be a point on the plane containing a triangle ABC. The lines MA, MB and MC intersect the lines BC, CA and AB at D, E and F respectively. The circumcircle of $\triangle DEF$ meets the lines BC, CA and AB respectively at D', E' and F'. Prove that AD', BE' and CF' are concurrent.



Using Ceva' Theorem, we have $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$. Also, $BD \cdot BD' = BF \cdot BF'$, $CE \cdot CE' = CD \cdot CD'$ and $AF \cdot AF' = AE \cdot AE'$. Thus

$$\frac{BD'}{BF'} \cdot \frac{CE'}{CD'} \cdot \frac{AF'}{AE'} = \frac{BF}{BD} \cdot \frac{CD}{CE} \cdot \frac{AE}{AF} = 1.$$

Thus $\frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = 1$. By the converse of Ceva's Theorem, AD', BE' and CF' are concurrent.

3. Let a, b, c, d be nonnegative real numbers. Show that

$$\left(\frac{abc + bcd + cda + dab}{4}\right)^2 \le \left(\frac{ab + bc + cd + da + ac + bd}{6}\right)^3.$$

[Remark: The proof I have is unsatisfying in many respects. You are welcome to contribute better proofs.]

First observe that the inequality to be shown is symmetric in the variables a, b, c, d and homogeneous. If $d = 0$, the inequality follows easily from AM-GM: $(abc)^{2/3} \le \frac{ab+ac+bc}{3}$. Otherwise, we may assume that $a \ge b \ge c \ge d = 1$. Denote by C the number $(abc)^{1/3}$. By our assumption, $C \ge 1$.

<u>Claim.</u> The function

$$f(x) = \left(\frac{x + C}{2}\right)^3 - \left(\frac{3x + C^3}{4}\right)^2$$

is increasing for $x \ge C^2$.

The only proof I have of the Claim is to use differential calculus to show that $f'(x) \ge 0$ for $x \ge C^2$.

Assuming the claim, we find that for $x \ge C^2$,

$$f(x) \ge f(C^2) = \frac{C^3}{16}(C^3 - 3C + 2) = \frac{C^3}{16}(C - 1)(C^2 + C - 2) \ge 0$$

since $C \ge 1$. By AM-GM inequality $\frac{ab+ac+bc}{3} \ge C^2$. Thus $f(\frac{ab+ac+bc}{3}) \ge 0$. Thus

$$\left(\frac{ab + ac + bc + C^3}{4}\right)^2 \le \left(\frac{ab + ac + bc + 3C}{6}\right)^3.$$

By AM-GM inequality, $3C \le a + b + c$. Hence

$$\left(\frac{ab + ac + bc + abc}{4}\right)^2 \le \left(\frac{ab + ac + bc + a + b + c}{6}\right)^3,$$

which is the inequality required (setting $d = 1$).

4. Let $ABCDEF$ be a convex hexagon inscribed in a circle. Assume that $AB = BC$, $CD = DE$ and $EF = FA$. Show that

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \ge \frac{3}{2}.$$

Consider the cyclic quadrilateral $ACEF$. By Ptolemy's Theorem,

$$AC \cdot EF + CE \cdot FA = AE \cdot FC.$$

Dividing by $FC(AC+CE)$ and using the fact that $EF = FA$, we have

$$\frac{FA}{FC} = \frac{AE}{AC+CE}.$$

Similarly,

$$\frac{BC}{BE} = \frac{AC}{CE+AE}$$
$$\frac{DE}{DA} = \frac{CE}{AE+AC}.$$

Let $AC = a$, $CE = b$ and $AE = c$. Then

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} = \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}.$$

By AM-HM inequality,

$$\frac{3}{\frac{1}{b+c} + \frac{1}{a+b} + \frac{1}{a+c}} \le \frac{b+c+a+b+a+c}{3}.$$

Hence

$$\frac{9}{2} \le (a+b+c)(\frac{1}{b+c} + \frac{1}{a+b} + \frac{1}{a+c}) = \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} + 3.$$

The desired inequality follows.

5. Let k be a given positive integer. Find the smallest n in terms of k so that for any set A of n integers, there are always two elements in A whose sum or difference is divisible by $2k$.

Consider a set of n integers and list their values mod $2k$ as $r_1, r_2, \ldots, r_n$. In particular, $0 \le r_i \le 2k-1$ for each i. The numbers $\{0, \ldots, 2k-1\}$ can be formed into $k+1$ groups

$$\{0\}, \{1, 2k-1\}, \ldots, \{k-1, k+1\}, \{k\}.$$

If $n \ge k+2$, there are i, j, $i \ne j$, so that r_i and r_j are in the same group. Then either $r_i + r_j$ or $r_i - r_j = 0 \mod 2k$.
On the other hand, if $n \le k+1$, we consider the set $\{0, 1, \ldots, k\}$. For any two elements, neither the sum nor the difference is divisible by $2k$.
Thus the smallest n satisfying the condition of the problem is $k+2$.