CWMI 2017

Day 1

1. Let $p$ be a prime and $n$ be a positive integer such that $p^2$ divides $\prod_{k=1}^{n} (k^2 + 1)$. Show that $p < 2n$.

2. Let $n$ be a positive integer such that there exist positive integers $x_1, x_2, \ldots, x_n$ satisfying
   \[ x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n) = 100n. \]
   Find the greatest possible value of $n$.

3. In triangle $ABC$, let $D$ be a point on $BC$. Let $I_1$ and $I_2$ be the incenters of triangles $ABD$ and $ACD$ respectively. Let $O_1$ and $O_2$ be the circumcenters of triangles $AI_1D$ and $AI_2D$ respectively. Let lines $I_1O_2$ and $I_2O_1$ meet at $P$. Show that $PD \perp BC$.

4. Let $n$ and $k$ be given integers such that $n \geq k \geq 2$. Alice and Bob play a game on an $n$ by $n$ table with white cells. They take turns to pick a white cell and color it black. Alice moves first. The game ends as soon as there is at least one black cell in every $k$ by $k$ square after a player moves, who is declared the winner of the game. Who has the winning strategy?

Day 2

5. Let $a_1, a_2, \ldots, a_9$ be 9 positive integers (not necessarily distinct) satisfying: for all $1 \leq i < j < k \leq 9$, there exists $l (1 \leq l \leq 9)$ distinct from $i, j$ and $j$ such that $a_i + a_j + a_k + a_l = 100$. Find the number of 9-tuples $(a_1, a_2, \ldots, a_9)$ satisfying the above conditions.

6. In acute triangle $ABC$, let $D$ and $E$ be points on sides $AB$ and $AC$ respectively. Let segments $BE$ and $DC$ meet at point $H$. Let $M$ and $N$ be the midpoints of segments $BD$ and $CE$ respectively. Show that $H$ is the orthocenter of triangle $AMN$ if and only if $B, C, E, D$ are concyclic and $BE \perp CD$.

7. Let $n = 2^\alpha \cdot q$ be a positive integer, where $\alpha$ is a nonnegative integer and $q$ is an odd number. Show that for any positive integer $m$, the number of integer solutions to the equation $x_1^2 + x_2^2 + \cdots + x_n^2 = m$ is divisible by $2^{\alpha+1}$.

8. Let $a_1, a_2, \ldots, a_n > 0$ ($n \geq 2$). Prove that
   \[ \sum_{i=1}^{n} \max\{a_1, a_2, \ldots, a_i\} \cdot \min\{a_i, a_{i+1}, \ldots, a_n\} \leq \frac{n}{2\sqrt{n-1}} \sum_{i=1}^{n} a_i^2. \]