Problem 1. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise,} \end{cases}$$

for each $n \geq 0$.

Determine all values of $a_0$ for which there is a number $A$ such that $a_n = A$ for infinitely many values of $n$.

Problem 2. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers $x$ and $y$,

$$f (f(x)f(y)) + f(x + y) = f(xy).$$

Problem 3. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit’s starting point, $A_0$, and the hunter’s starting point, $B_0$, are the same. After $n - 1$ rounds of the game, the rabbit is at point $A_{n-1}$ and the hunter is at point $B_{n-1}$. In the $n^{th}$ round of the game, three things occur in order.

(i) The rabbit moves invisibly to a point $A_n$ such that the distance between $A_{n-1}$ and $A_n$ is exactly 1.

(ii) A tracking device reports a point $P_n$ to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between $P_n$ and $A_n$ is at most 1.

(iii) The hunter moves visibly to a point $B_n$ such that the distance between $B_{n-1}$ and $B_n$ is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after $10^9$ rounds she can ensure that the distance between her and the rabbit is at most 100?
Problem 4.  Let $R$ and $S$ be different points on a circle $\Omega$ such that $RS$ is not a diameter. Let $\ell$ be the tangent line to $\Omega$ at $R$. Point $T$ is such that $S$ is the midpoint of the line segment $RT$. Point $J$ is chosen on the shorter arc $RS$ of $\Omega$ so that the circumcircle $\Gamma$ of triangle $JST$ intersects $\ell$ at two distinct points. Let $A$ be the common point of $\Gamma$ and $\ell$ that is closer to $R$. Line $AJ$ meets $\Omega$ again at $K$. Prove that the line $KT$ is tangent to $\Gamma$.

Problem 5.  An integer $N \geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following $N$ conditions hold:

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players,

\vdots

(N) no one stands between the two shortest players.

Show that this is always possible.

Problem 6.  An ordered pair $(x, y)$ of integers is a primitive point if the greatest common divisor of $x$ and $y$ is 1. Given a finite set $S$ of primitive points, prove that there exist a positive integer $n$ and integers $a_0, a_1, \ldots, a_n$ such that, for each $(x, y)$ in $S$, we have:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$