41st International Mathematical Olympiad

Taejon, Korea, July 2000.

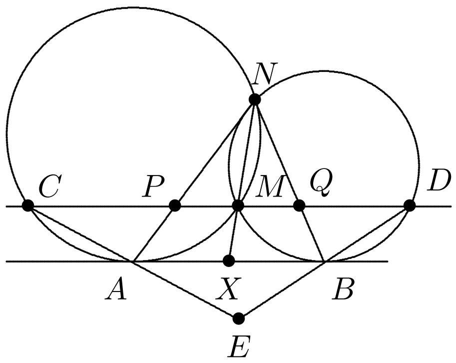
1. Two circles Γ_1 and Γ_2 intersect at M and N.

Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again C and the circle Γ_2 at D.

Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q.

Show that $EP = EQ$.

Soln. (Official solution): M is in fact the midpoint of PQ. To see this, extend NM meeting AB at X. Then X is the midpoint of the common tangent AB, because X being on the radical axis MN is of equal power to the two circles. As PQ is parallel to AB, M is the midpoint of PQ.



An easy diagram chasing of the angles shows that triangle EAB is congruent to triangle MAB. Hence EM is perpendicular to AB, thus perpendicular to PQ. From this it follows that $EP = EQ$.

2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Soln. (Official solution): Write $a = x/y$, $b = y/z$ and $c = z/x$ for some positive numbers x, y, z. Rewriting the inequality in terms of x, y, z we have

$$(x - y + z)(y - z + x)(z - x + y) \leq xyz.$$

Let the three factors on the left hand side be u, v, w, respectively. Since any two of u, v, w have positive sum, at most one of them is negative. If exactly one of u, v, w is negative, then the inequality holds. We are left with the case $u, v, w > 0$. By the AM-GM inequality, we have

$$\sqrt{uv} \leq \frac{u+v}{2} = x.$$

Likewise $\sqrt{vw} \leq y$, $\sqrt{wu} \leq z$. Hence $uvw \leq xyz$ as desired.

Second soln. We may assume, without loss of generality, that $a \geq 1 \geq c (> 0)$. Let $d = 1/c$. Then $d \geq 1$. Substituting $b = 1/(ac)$ into the right hand side of the inequality and multiplying it out, we have

$$\begin{aligned}&(a+1/a)+(d+1/d)+(a/d+d/a)-(ad+d/(a^2)+a/(d^2))-2\\ &=(a-1)(1-d)+(a-d)(d-1)/(d^2)+(d-a)(a-1)/(a^2)+1.\end{aligned}$$

We may assume $a \geq d$. Then

$$(d-a)(a-1)/(a^2) \leq 0.$$

The first two terms can be combined to get

$$\frac{(d-1)(-ad^2+d^2+a-d)}{d^2} = \frac{(d-1)^2(d-ad-a)}{d^2} \leq 0.$$

So the whole expression is ≤ 1.

Third soln. Denote the left hand side of the inequality by L. If $a-1+1/b < 0$, then $a < 1$ and $b > 1$. Thus $b-1+1/c$ and $c-1+1/a$ are both positive, whence L is negative and the inequality holds. The same argument applies when one of the other two factors is negative. Hence forth we assume that all the three factors in L are positive. Note that $abc = 1$ implies $b(a-1+1/b) = (1/c-b+1)$, $c(b-1+1/c) = (1/a-c+1)$, $a(c-1+1/a) = (1/b-a+1)$. Thus $L = (a+1-1/b)(b+1-1/c)(c+1-1/a)$ and

$$L^2 = \big(a^2-(1-1/b)^2\big)\big(b^2-(1-1/c)^2\big)\big(c^2-(1-1/a)^2\big).$$

All these imply that

$$\begin{aligned}0 \leq a^2-(1-1/b)^2 \leq a^2,\\ 0 \leq b^2-(1-1/c)^2 \leq b^2,\\ 0 \leq c^2-(1-1/a)^2 \leq c^2\end{aligned}$$

which in turn implies that

$$L^2 \leq (abc)^2 = 1 \quad \text{and} \quad L \leq 1.$$

3. Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at the same point.

For a positive real number λ, define a *move* as follows:

choose any two fleas, at points A and B, with A to the left of B;

let the flea at A jump to the point C on the line to the right of B with $BC/AB = \lambda$.

Determine all values of λ such that, for any point M on the line and any initial positions of the n fleas, there is a finite sequence of moves that will take all the fleas to the right of M.

Soln. (Official solution): We adopt the strategy to let leftmost flea jump over the rightmost flea. After k moves, let d_k denote the distance of the leftmost and the rightmost flea and δ_k denote the minimum distance between neighbouring fleas. Then $d_k \geq (n-1)\delta_k$.

After the $(k+1)$st move, there is a new distance between neighbouring fleas, namely λd_k. It can be the new minimum distance, so that $\delta_{k+1} = \lambda d_k$; and if not, then certainly $\delta_{k+1} \geq \delta_k$. In any case

$$\frac{\delta_{k+1}}{\delta_k} \geq \min\left\{1, \frac{\lambda d_k}{\delta_k}\right\} \geq \min\{1, (n-1)\lambda\}.$$

Thus if $\lambda \geq 1/(n-1)$ then $\delta_{k+1} \geq \delta_k$ for all k; the minimum distance does not decrease. So the positive of the leftmost flea keeps on shifting by steps of size not less that a positive constant, so that, eventually all the fleas will be carried as far to the right as we please.

Conversely, if $\lambda < 1/(n-1)$, we'll prove that for any initial configuration, there is a point M beyond which no flea can reach. The position of the fleas will be viewed as real numbers. Consider an arbitrary sequence of moves. Let s_k be the sum of all the numbers representing the positions of he fleas after the kth move and let w_k be the greatest of these numbers (i.e. the position of the rightmost flea). Note that $s_k \leq nw_k$. We are going to show that the sequence (w_k) is bounded.

In the $(k+1)$st move a flea from a A jumps over B, landing at C; let these points be represented by the numbers a, b, c. Then $s_{k+1} - s_k + c - a$.

By the given rules, $c - b = \lambda(b - a)$; equivalently $\lambda(c - a) = (1 + \lambda)(c - b)$. Thus

$$s_{k+1} - s_k = c - a = \frac{1+\lambda}{\lambda}(c - b).$$

Suppose that $c > w_k$; the flea that has just jumped took the new rightmost position $w_{k+1} = c$. Since b was the position of some flew after the kth move, we have $b \leq w_k$ and

$$s_{k+1} - s_k = \frac{1+\lambda}{\lambda}(c - b) \geq \frac{1+\lambda}{\lambda}(w_{k+1} = w_k).$$

This estimate is valid also when $c \leq w_k$, in which case $w_{k+1} - w_k = 0$ and $s_{k+1} - s_k = c - a > 0$.

Consider the sequence of numbers

$$z_k = \frac{1+\lambda}{\lambda} w_k - s_k \quad \text{for} \quad k = 0, 1, \ldots.$$

The estimate we have just worked out shows that $z_{k+1} - z_k \le 0$; the sequence is nonincreasing, and consequently $z_k \le z_0$ for all k.

We have assume that $\lambda < 1/(n-1)$. Then $1 + \lambda > n\lambda$, and we can write

$$z_k = (n+\mu)w_k - s_k, \quad \text{where} \quad \mu = \frac{1+\lambda}{\lambda} - n > 0.$$

So we get the inequality $z_k = \mu w_k + (nw_k - s_k) \ge \mu w_k$. It follows that $w_k \le z_0/\mu$ for all k. Thus the position of the rightmost flea never exceeds a constant (depending on n, λ and the initial configuration, but not on the strategy of moves). In conclusion, the values of λ, asked about, are all real numbers not less than $1/(n-1)$.

4. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card.

A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen.

How many ways are there to put all the cards into the boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

Soln. (Official solution): Suppose $1, 2 \ldots, k$, $k \ge 2$, are in box 1, and $k+1$ in box 2 and m is the smallest number in box 3. Then $m-1$ is either in box 1 or 2. But it can't be in box 1 for $(m)+(k) = (m-1)+(k+1)$, but it can't be in box 2 either as $(m)+(1) = (m-1)+2$. Thus we conclude that 1 and 2 are in different boxes. So we assume that 1 is in box 1, and $2, \ldots, k$, $k \ge 2$ are in box 2, $k+1$ not in box 2 and m is the smallest number in box 3. If $m > k+1$, then $k+1$ is in box 1. Also $m-1$ is not in box 1 as $(m-1)+(2) = (m)+(1)$. Thus $m-1$ is in box 2. This is not possible as $(m)+(k) = (m-1)+(k+1)$. Thus $m = k+1$. If $k = 2$, we have 1, 2, 3 in different boxes. Since a in box 1, $a+1$ in box 2, $a+2$ box 3 imply that $a+3$ is in box 1. We have box i contains all the numbers congruent to $i \pmod 3$. This distribution clearly works since $a \equiv i, b \equiv j \pmod 3$ imply $a+b \equiv k \pmod 3$ where $k \not\equiv i, j \pmod 3$.

Now suppose that $k \ge 3$. We conclude that $k+2$ can't be included in any box. Thus $k = 99$. We see that this distribution also works.

Hence there are altogether 12 ways.

Second soln. Consider 1, 2 and 3. If they are in different boxes, then 4 must be in the same box as 1, 5 in the same box as 2 and so on. This leads to the solution where all numbers congruent to each other mod 3 are in the same box.

Suppose 1 and 2 are in box 1 and 3 in box 2. Then 4 must be in box 1 or 2. In general, if $k(\ge 4)$ is in either box 1 or 2, then $k+1$ also must be in box 1 or 2. Thus box 3 is empty which is impossible.

Similarly, it is impossible for 1 and 3 to be in box 1 and 2 in box 2.

Thus we are left with the case where 1 is in box 1 and 2 and 3 in box 2. Suppose box 2 contains $2, \ldots k$, where $k \ge 3$, but does not contain $k+1$ and m is the smallest number in box 3. Then $m > k$. If $m > k+1$, then $k+1$ must be box 1 and we see that no box can contain $m-1$. Thus $m = k+1$. If $k < 99$, we see that no box can contain $k+2$. Thus $k = 99$. It is easy to see that this distribution works. Thus there altogther 12 ways.

Third soln. (official): We show that the answer is 12. Let the colour of the number i be the colour of the box which contains it. In the sequel, all numbers considered are assumed to be integers between 1 and 100.

Case 1. There is an i such that i, $i+1$, $i+2$ have three different colours, say **rwb**. Then, since $i + (i+3) = (i+1) + (i+2)$, the colour of $i+3$ can be neither **w**(the colour of $i+1$) nor **b**(the colour of $i+2$). It follows that $i+3$ is **r**. Using the same argument, we see that the next numbers are also **rwb**. In fact the argument works backwards as well: the previous three numbers are also **rwb**. Thus we have 1, 2 and 3 in different boxes and two numbers are in the same box if there are congruent mod 3. Such an arrangement is good as $1+2$, $2+3$ and $1+3$ are all different mod 3. There are 6 such arrangements.

Case 2. There are no three neighbouring numbers of different colours. Let 1 be red. Let i be the smallest non-red number, say white. Let the smallest blue number be k. Since there is no **rwb**, we have $i+1 < k$.

Suppose that $k < 100$. Since $i + k = (i-1) + (k+1)$, $k+1$ should be red. However, in view if $i + (k+1) = (i+1) + k$, $i+1$ has to be blue, which draws a contradiction to the fact that the smallest blue is k. This implies that k can only be 100.

Since $(i-1) + 100 = i + 99$, we see that 99 is white. We now show that 1 is red, 100 is blue, all the others are white. If $t > 1$ were red, then in view of $t + 99 = (t-1) + 100$, $t-1$ should be blue, but the smallest blue is 100.

So the colouring is **rww**...**wwb**, and this is indeed good. If the sum is at most 100, then the missing box is blue; if the sum is 101, then it is white and if the sum is greater than 101, then it is red. The number of such arrangements is 6.

5. Determine whether or not there exists n such that

> n is divisible by exactly 200 different prime numbers and $2^n + 1$ is divisible by n.

Soln. (oficial): The answer is yes and we shall prove it proving a more general statement: *For each $k \in \mathbb{N}$, there exists $n = n(k) \in \mathbb{N}$ such that $n \mid 2^n + 1$, $3 \mid n$ and n has exactly k prime factors.* We shall prove it by induction on k.

We have $n(1) = 3$. We then assume for some $k \geq 1$, there exists $n = n(k)$ with the desired properties. Then n is odd. Since $2^{3n} + 1 = (2^n + 1)(2^{2n} - 2^n + 1)$ and 3 divides the second factor, we have $3n \mid 2^{3n} + 1$. For any positive odd integer m, we have $2^{3n} + 1 \mid 2^{3nm} + 1$. Thus if p is prime number such that $p \nmid n$ and $p \mid 2^{3n} + 1$, then $3np | 2^{3np} + 1$ and $n(k+1) = 3pn$ has the desired properties. Thus the proof would be complete if we can find such a p. This is achieved by the following lemma:

> **Lemma.** For any integer $a > 2$ such that $3 \mid a + 1$, there exists a prime number p such that $p \mid a^3 + 1$ but $p \nmid a + 1$.

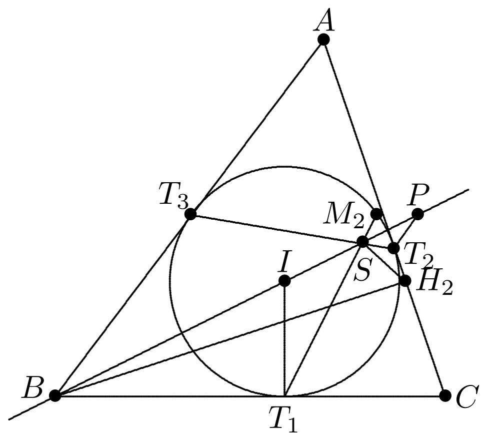
Proof. Assume that this is false for a certain integer $a > 2$. Since $a^3+1 = (a+1)(a^2-a+1)$, each prime divisor of a^2-a+1 divides $a+1$. Since $a^2-a+1 = (a+1)(a-2)+3$, we conclude that a^2-a+1 is a power of 3. Since $a+1$ and $a-2$ are both multiples of 3, we conclude that $9 \nmid a^2-a+1$. This gives a contradiction as $a^2-a+1 > 3$ for $a > 2$.

6. Let AH_1, BH_2, CH_3 be the altitudes of an acute-angled triangle ABC. The incircle of the triangle ABC touches the sides BC, CA, AB at T_1, T_2, T_3, respectively. Let the lines ℓ_1, ℓ_2, ℓ_3 be the reflections of the lines H_2H_3, H_3H_1, H_1H_2 in the lines T_2T_3, T_3T_1, T_1T_2, respectively.

Prove that ℓ_1, ℓ_2, ℓ_3 determine a triangle whose vertices lie on the incircle of the triangle ABC.

Soln. (Official solution): Let M_1, M_2, M_3 be the reflections of T_1, T_2, T_3 across the bisectors of $\angle A, \angle B, \angle C$, respectively. The points M_1, M_2, M_3 obviously lie of the incircle of $\triangle ABC$. We prove that they are the vertices of the triangle formed by the images in question, which settle the claim.

By symmetry, it suffices to show that the reflection l_1 of H_1H_2 in T_2T_3 passes through M_2. Let I be the incentre of $\triangle ABC$. Note that T_2 and H_2 are always on the same side of BI, with T_2 closer to BI than H_2. We consider only the case when C is on this same side of BI, as in the figure (minor modifications are needed if C is on the other side).



Let $\angle A = 2\alpha, \angle B = 2\beta, \angle C = 2\gamma$.

Claim 1: the mirror image of H_2 with respect to T_2T_3 lies on the line BI.

Proof of claim 1: Let $\ell \perp T_2T_3$, $H_2 \in \ell$. Denote by P and S the points of intersection of BI with ℓ and BI with T_2T_3. Note that S lies on both line segments T_2T_3 and BP. It is sufficient to prove that $\angle PSH_2 = 2\angle PST_2$. We have $\angle PST_2 = \angle BST_3$ and by the external angle theorem,

$$\angle BST_3 = \angle AT_3S - \angle T_3BS = (90^\circ - \alpha) - \beta = \gamma.$$

Next $\angle BST_1 = \angle BST_3 = \gamma$ by symmetry across BI. Note that C and S are on the same side of IT_1, since $\angle BT_1S = 90^\circ + \alpha > 90^\circ$. Then, in view of the equalities $\angle IST_1 = \angle ICT_1 = \gamma$, the quadrilateral SIT_1C is cyclic, so $\angle ISC = \angle IT_1C = 90^\circ$. Hence $\angle BSC =$

$\angle BH_2C$, and hence the quadrilateral BCH_2S is cyclic. It shows that $\angle PSH_2 = \angle C = 2\gamma = 2\angle PST_2$, as needed. This completes of the proof of the claim.

Note that the proof of the claim also gives

$$\angle BPT_2 = \angle SH_2T_2 = \beta,$$

by symmetry across T_2T_3 and because the quadrilateral BCH_2S is cyclic. Then, since M_2 is the reflection of T_2 across BI, we obtain $\angle BPM_2 = \angle BPT_2 = \beta = \angle CBP$, and so PM_2 is parallel to BC. To prove that M_2 lies on ℓ_1, it now suffices to show that ℓ_1 is also parallel to BC.

Suppose $\beta \neq \gamma$; let the line CB meet H_2H_3 and T_2T_3 at D and E, respectively. (Note that D and E lie on the line BC on the same side of the segment BC.) An easy angle computation gives $\angle BDH_3 = 2|\beta - \gamma|$, $\angle BET_3 = |\beta - \gamma|$, and so the line ℓ_1 is indeed parallel to BC. The proof is now complete.