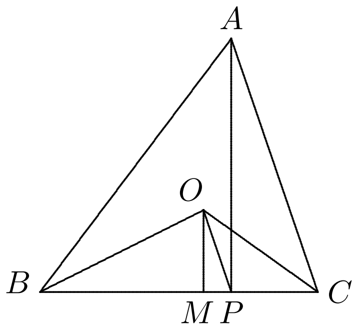
42nd International Mathematical Olympiad

Washington DC, United States of America, July 2001

1. Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A.

Suppose that $\angle BCA \geq \angle ABC + 30°$.

Prove that $\angle CAB + \angle COP < 90°$.



Essentially, the trick is to convert this to a trigonometry inequality. There are many ways to do it, we present the simplest one. Most of the other solutions involve proofing $PB \geq 3PC$ from the desired result follows readily.

Soln. Let R be the circumradius. Then

$$\begin{aligned} CP &= AC\cos C = 2R\sin B\cos C \\ &= R(\sin(B+C) - \sin(C-B)) \leq R(1-\sin(C-B)) \\ &\leq R(1-\sin 30°) = R/2. \end{aligned}$$

So, $OP > OC - PC \geq PC$, and whence $\angle PCO > \angle POC$. The desired result then follows from the fact that $\angle PCO + \angle CAB = 90°$.

Second soln. First we prove that $PB \geq 3PC$. We have $PB = AP\cot B$, $PC = AP\cot C$. Therefore $PB \geq 3PC$ if and only if $\tan C \geq 3\tan B$. Since $C \geq B + 30°$, and C is acute, we have $\tan C \geq \tan(B + 30°)$. Thus

$$\begin{aligned} \tan C - 3\tan B &\geq \tan(B+30°) - 3\tan B \\ &= \frac{\tan B + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}\tan B} - 3\tan B \\ &= \frac{3}{\sqrt{3} - \tan B}(\tan B - \tfrac{1}{\sqrt{3}})^2 \geq 0 \end{aligned}$$

since $B < 60°$. Thus $PB \geq 3PC$ and whence $PC \leq PM$ where M is the midpoint of BC. This, together with the fact that OP is the hypothenuse of the right-angled triangle OPM, implies $OP > PM \geq PC$.

2. Prove that

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$$

for all positive real numbers a, b and c.

Soln. Note that $f(x) = \frac{1}{\sqrt{x}}$ is convex for positive x. Recall weighted Jensen's inequality:-

$$af(x) + bf(y) + cf(z) \geq (a+b+c)f(ax+by+cz).$$

Apply this to get

$$\text{LHS} \geq \sqrt{\frac{(a+b+c)^3}{a^3+b^3+c^3+24abc}} \geq 1.$$

The last step follows because by the AM-GM inequality, we have

$$(a+b+c)^3 \geq a^3+b^3+c^3+24abc.$$

Second soln. By Cauchy-Schwarz Inequality we have

$$\text{LHS} \times \left(a\sqrt{a^2+8bc} + b\sqrt{b^2+8ac} + c\sqrt{c^2+8ab}\right) \geq (a+b+c)^2$$

and

$$\begin{aligned}
&(a\sqrt{a^2+8bc} + b\sqrt{b^2+8ac} + c\sqrt{c^2+8ab}) \\
&\qquad = \sqrt{a}\sqrt{a^3+8abc} + \sqrt{b}\sqrt{b^3+8abc} + \sqrt{c}\sqrt{c^3+8abc} \\
&\qquad \leq \sqrt{a+b+c}\sqrt{a^3+b^3+c^3+24abc} \\
&\qquad \leq (a+b+c)^2.
\end{aligned}$$

The inequality thus follows.

Third soln. Let $abc = 1$. Then divide numerator and denominator by a in the first term, b in second, and c in the third and then substitute $x = 1/a^3$, $y = 1/b^3$, $z = 1/c^3$ with $xyz = 1$. The left hand side becomes.

$$\frac{1}{\sqrt{1+8x}} + \frac{1}{\sqrt{1+8y}} + \frac{1}{\sqrt{1+8z}}.$$

Let the denominators be u, v and w, respectively. Then the given inequality is equivalent to

$$(uv + uw + vw) \geq uvw.$$

with u, v, w all positive. Upon squaring both sides, the inequality is equivalent to

$$1 + 4(x + y + z) + uvw(u + v + w) \geq 256.$$

This follows from $x+y+z \geq 3$, $uvw(u+v+w) \geq 3(uvw)^{4/3} = 3[(1+8x)(1+8y)(1+8z)]^{2/3} \geq 243$.

Fourth soln. Without loss of generality, we can assume that $a \geq b \geq c$. Let $A = \sqrt{a^2 + b^2 + c^2 + 6bc}$, $B = \sqrt{a^2 + b^2 + c^2 + 6ac}$, $C = \sqrt{a^2 + b^2 + c^2 + 6ab}$. Then $A \leq B \leq C$. By squaring both sides, simplify and the using AM-GM, we have

$$A + B + C \leq 3(a + b + c)$$

and

$$\sqrt{a^2 + 8bc} \leq A, \quad \sqrt{b^2 + 8ac} \leq B, \quad \sqrt{c^2 + 8ab} \leq C.$$

Thus we have

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ac}} \frac{c}{\sqrt{c^2 + 8ab}} \geq \frac{a}{A} + \frac{b}{B} + \frac{c}{C}$$

and by Chebyschev's inequality, we have

$$\left(\frac{a}{A} + \frac{b}{B} + \frac{c}{-} \right) (A + B + C) \geq 3(a + b + c).$$

Thus

$$\frac{a}{A} + \frac{b}{B} + \frac{c}{C} \geq \frac{3(a + b + c)}{A + B + C} \geq 1.$$

Fifth soln. (official solution.) First we shall prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} \geq \frac{a^{4/3}}{a^{4/3} + b^{4/3} + c^{4/3}},$$

or equivalently, that

$$\left(a^{4/3} + b^{4/3} + c^{4/3}\right)^2 \geq a^{2/3}(a^2 + 8bc),$$

or equivalently, that

$$b^{4/3} + c^{4/3} + 2a^{4/3}b^{4/3} + 2a^{4/3}c^{4/3} + 2b^{4/3}c^{4/3} \geq 8a^{2/3}bc.$$

The last inequality follows from the AM-GM inequality. Similarly, we have

$$\frac{b}{\sqrt{b^2 + 8ac}} \geq \frac{b^{4/3}}{a^{4/3} + b^{4/3} + c^{4/3}}, \qquad \frac{c}{\sqrt{c^2 + 8ab}} \geq \frac{c^{4/3}}{a^{4/3} + b^{4/3} + c^{4/3}}.$$

The result then follows by adding these three inequalities.

3. Twenty-one girls and twenty-one boys took part in a mathematical contest.

Each contestant solved at most six problems.

For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

Note. One useful way to investigate this problem is to form an incidence matrix. Let $B_1, B_2, \ldots, B_{21}$ be the boys and $G_1, \ldots, G_{21}$ be the girls and $P_1, \ldots, P_n$ be the problems. Set up an incidence matrix with the columns indexed by the problems and the rows indexed by the students. The entry at (S, P_i) is 1 if S solves P_i and 0 otherwise. We present two solutions based on this incidence matrix.

Soln. Let b_i be the number boys who solve P_i and g_i be the number of girls who wolve P_i. Then the number of ones in every row is at most 6. Thus $\sum_{i=1}^n b_i \le 6|B|$ and $\sum_{i=1}^n g_i \le 6|G|$.

In this matrix the rows B_i and G_j have at least a pair of ones in the same column because every boy and every girl solve a common problem. Call such a pair of ones a one-pair. Thus the number of one-pairs is at least 21^2. However, counting by the columns, the number of one-pairs is $\sum b_i g_i$. Thus we have

$$\sum g_i b_i \ge 21^2.$$

Now suppose that the conclusion is false. Then $b_i \ge 3$ implies $g_i \le 2$ and vice versa. Let P_G be the set of problems, each of which is solved by at least 3 girls and at most 2 boys, P_B be the set of problems, each solved by at least 3 boys and at most 2 girls and P_X be the set of problems, each of which is solved by at most 2 boys and at most 2 girls. Thus

$$\sum b_i g_i = \sum_{P_i \in P_B} b_i g_i + \sum_{P_i \in P_G \cup P_X} b_i g_i \le 2 \sum_{P_i \in P_B} b_i + 2 \sum_{P_i \in P_G \cup P_X} g_i.$$

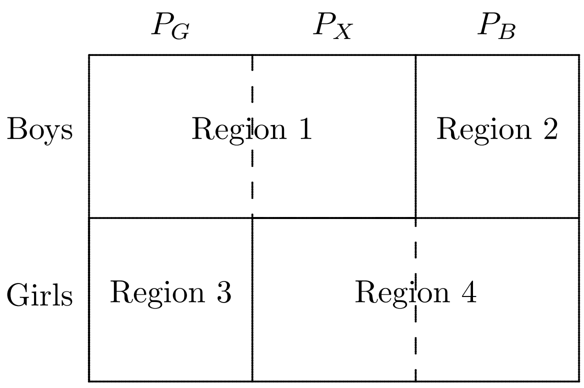
Now for any girl G_i, consider the matrix M_i with whose columns correspond to problems solved by G_i and whose rows are all the boys. Then in this matrix, every row has at least a one. Thus there are at least 21 ones in this matrix. By the pigeonhole principle, there is a column, say P_j with at least 4 ones. Thus each girl solves at least one problem in P_B. Hence $\sum_{P_i \in P_B} g_i \ge |G|$ or equivalently, $\sum_{P_i \in P_G \cup P_X} g_i \le 5|G|$. Similarly, $\sum_{P_i \in P_B} b_i \le \sum_{P_i \in P_B \cup P_X} b_i \le 5|B|$. Thus we have

$$21^2 \le \sum b_i g_i \le 10(|G| + |B|) = 420$$

a contradiction.

Second soln. With the same notation as in the first solution, divide the incidence matrix M into two part: M_B which is formed by the columns in $P_B \cup P_X$ and M_G which is formed by the columns in P_G. The matrix M has 441 one-pairs. Thus one of these two submatrices, say M_B, has at least 221 one-pairs. (The case for M_G foolows by symmetry.) Thus one of the girls, say G_1, contributes at least 11 one-pairs in M_B. Since each one in row G_1 contributes at most 2 one-pairs in M_B, there are 6 ones in row G_1 in M_B. This means the row G_1 in M_G does not have any ones. Thus G_1 contributes at most 12 one-pairs in M. But G_1 should contribute at least 21 one-pairs and we have a contridiction.

Third soln. Suppose on the contrary that no problem was solved by at least three girls and at least three boys. With P_G, P_B, P_X defined as in the first solution, we arrange the problems as shown in the figure.



Consider any girl G_i. She contributes at least 21 one-pairs. As in the first solution, she solves at least 1 problem in P_B and at most 5 problems in $P_B \cup P_X$. (Since there are 21 girls, there must be at least $\lceil \frac{21}{2} \rceil = 11$ questions in P_B. By a similar argument using the boys, there must be at least 11 questions in P_G as well. We don't need this in this solution. But this fact is used in the third solution and is obtained in a different way there.) Thus the number of ones in Region 3 is at least 21. Similarly, the number of ones in Region 2 is at least 21. The girl G_i contributes at least 21 ones in Regions 1 and 2 since she is associated with 21 one-pairs. At most 10 of these ones are in Region 2 and therefore at least 11 are in Region 1. So the girls contribute $21 \times 11 = 231$ ones in Region 1, counting repetition. Each problem in P_G is solved by at most 2 girls. Thus the total number of ones in Region 1 (without repetition) is at least $\lceil 231/2 \rceil = 116$. Likewise, the total number of ones in Region 4 is at least 116. So the total number of ones in the matrix is at least $116 + 116 + 21 + 21 = 274$ contradicting the fact that the total number of ones is at most $42 \times 6 = 252$.

Fourth soln. Suppose each problem P_i is solved by g_i girls and b_i boys. Then $\sum g_i b_i \geq 21^2 = 441$ since each boy and each girl solved a common problem. We assume that the conclusion is false, i.e. $\min\{g_i, b_i\} \leq 2$. We also assume that each problem is solved by at least one boy and at least one girl. So

$$g_i + b_i \geq \frac{g_i b_i}{2} + 1.5 \quad \text{and} \quad \sum_{i=1}^{n} g_i + b_i \geq 220.5 + 1.5n.$$

Since each boy and each girl solved at most 6 problems, we have $\sum g_i + b_i \leq 6 \times 21 \times 2 = 252$. From these we have $n \leq 21$.

Now consider a 21×21 grid, with one side representing girls, the other boys. Each cell in the grid is filled with the problems solved by both the corresponding boy and girl. There are at most 6 problems in each row and each column and each cell must contain at least one problem. In each row R_i there is problem P_i that appears at least three times. Similarly, each column C_j has such a problem P'_j. If $P_i = P'_j$ for some i, j, then this problem is solved by three boys and three girls. So we assume that $\{P_i\}$ and $\{P'j\}$ are disjoint. Also if there exist i, j, k such that $P_i = P_j = P_k$, the this problem is solved by three girls and three boys. So the set $\{P_i\}$ contains at least 11 problems. Similarly, the set $\{P'_j\}$ contains at least 11 problems. Thus there are at least 22 problems, a contradiction.

4. Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i a_i.$$

Prove that there are two permutations b and c, $b \neq c$, such that $n!$ is a divisor of $S(b)-S(c)$.

Soln. *Official solution. This is the standard double counting argument.* Compute the sum $\sum S(a)$, over all permutations. For each $i = 1, 2, \ldots, n$, the term $k_j i$, $j = 1, 2, \ldots, n$, appears $(n-1)!$ times. Thus its contribution to $\sum S(a)$ is $(n-1)! k_j i$. Thus

$$\sum S(a) = (n-1)! \sum_i i \sum_j k_j = \frac{(n+1)!}{2} \sum_j k_j \qquad (*)$$

Now suppose that the conclusion is false. Then the set $\{S(a)\}$ is a complete set of residues mod $n!$. Thus

$$\sum S(a) \equiv 1 + 2 + \cdots + n! = \frac{(n!+1)n!}{2} \equiv \frac{n!}{2} \not\equiv 0 \pmod{n!}.$$

But from $(*)$, we have $\sum S(a) = n![(n+1)/2] \sum k_j \equiv 0 \pmod{n!}$. (Note $(n+1)/2$ is an integer as n is odd.) Thus we have a contradiction.

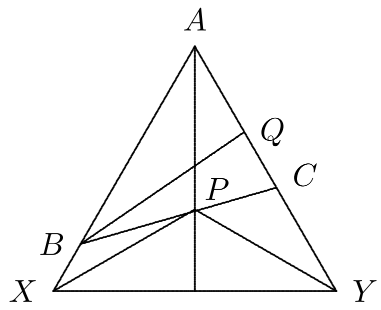
5. In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA.

It is known that $\angle BAC = 60°$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle ABC?

Soln. Extend AB to X such that $BX = BP$. Similarly, let Y be the point on AC (extended if necessary) on the opposite side of Q as A such that $BQ = QY$. Since

$AB + BP = AQ + QB$, this implies that $AX = AY$ by construction, and hence ΔAXY is equilateral with AP being the perpendicular bisector of XY.



We consider first the case where Y does not coincide with C and lies on AC extended (as in the figure). Let $\angle ABQ = \angle CBQ = x$. Then since $BX = BP$, $\angle BXP = \angle BPX = x$. Also, $\angle BQC = 60^\circ + x$ and $BQ = QY$ imply that $\angle QBY = \angle QYB = 60^\circ - \frac{x}{2}$, so $\angle PBY = 60^\circ - \frac{3x}{2}$. Since AP is the perpendicular bisector of XY, $\angle PXY = \angle PYX$, so that $\angle PYC = \angle PXB = x$. Thus, $\angle PYB = \angle QYB - x = 60^\circ - \frac{3x}{2}$. Hence $\angle PBY = \angle PYB$ and $PB = PY = PX$, which implies that ΔPBX is equilateral and $x = 60^\circ$. However, this is a degenerate case since $\angle BAC = 60^\circ$ and $\angle ABC = 2x = 120^\circ$. The case where Y does not coincide with C and lies in the interior of AC is similar, except that this time $\angle PBY = \angle PYB = \frac{3x}{2} - 60^\circ$. We once again reach the conclusion that ΔPBX is equilateral and $x = 60^\circ$, so this is a degenerate case once again.

This leaves just one case to consider where Y coincides with C. In this case, $BQ = QC$ and so $\angle ABQ = \angle CBQ = \angle BCQ = \frac{180^\circ - 60^\circ}{3} = 40^\circ$. We can verify that this 40°-60°-80° triangle verifies the condition of the question: Extend AB to X so that $BX = BP$. Then ΔAPX is congruent to ΔAPC, since $\angle PXB = \angle ACB = 40^\circ$, $\angle BAP = \angle CAP = 30^\circ$ and AP is a common side. It follows that $PX = PC$ and so $\angle PXC = \angle PCX = 20^\circ$. Hence, $\angle AXC = \angle ACX = 60^\circ$, so ΔAXC is equilateral. Thus, $AX = AC \;\Rightarrow\; AB + BX = AQ + QY \;\Rightarrow\; AB + BP = AQ + QB$. QED.

Second soln. Let $\angle ABQ = \angle CBQ = \alpha$, $AB = c$. $AC = b$, $BC = a$. By sine rule, we have

$$BP = \frac{c \sin 30^\circ}{\sin(2\alpha + 30^\circ)}.$$

By sine rule again, we have

$$BQ = \frac{c \sin 60^\circ}{\sin(\alpha + 60^\circ)}, \quad AQ = \frac{c \sin \alpha}{\sin(\alpha + 60^\circ)}.$$

From $AB + BP = AQ + QB$, we have

$$1 + \frac{1}{2\sin(2\alpha + 30^\circ)} = \frac{\sin 60^\circ + \sin\alpha}{\sin(\alpha + 60^\circ)}.$$

Let $\alpha = \beta + 30^\circ$, we get

$$1 + \frac{1}{2\sin(2\beta + 90^\circ)} = \frac{\sin 60^\circ + \sin(\beta + 30^\circ)}{\sin(\beta + 90^\circ)}$$

$$\text{i.e.} \quad \frac{\cos\beta}{\cos 2\beta} = \sqrt{3} + \sqrt{3}\sin\beta - \cos\beta.$$

Let $\sin\beta = x$. Since $\alpha < 90°$, $\beta < 60°$, $x \neq \pm 1$. We get the equation:

$$\frac{\sqrt{1-x^2}}{1-2x^2} + \sqrt{1-x^2} = \sqrt{3}(1+x)$$
$$\text{i.e.} \qquad (2x-1)(8x^3-6x+1) = 0$$

$x = 1/2$ implies $\beta = 30°$, $\alpha = 60°$. Therefore the angles of the triangle are $120, 60, 0$ which is impossible. Thus we have $8\sin^3\beta - 6\sin\beta + 1 = 0$. This implies $\sin 3\beta = -1/2$, i.e., $\beta = 70°$, $\alpha - 40°$. Thus the angles are $80, 40, 60$.

6. Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$ac + bd = (b+d+a-c)(b+d-a+c).$$

Prove that $ab + cd$ is not prime.

Soln. Write the original condition as

$$a^2 - ac + c^2 = b^2 + bd + d^2 \qquad (*)$$

Assume that $ab + cd = p$ is prime. Then $a = (p - cd)/b$. Substituting this into $(*)$, we get

$$p(ab - cd - cb) = (b^2 - c^2)(b^2 + bd + d^2).$$

since $1 < b^2 - c^2 < ab < p$, we have $p \mid (b^2 + bd + d^2)$. But

$$b^2 + bd + d^2 < 2ab + cd < 2p,$$

we have $b^2 + bd + d^2 = p$. Hence, by equating the expressions for p, we get

$$b(b+d-a) = d(c-d).$$

Since $\gcd(b, d) = 1$, we have $b \mid (c-d)$, a contradiction because $0 < c - d < b$.

Second soln. *official solution.* Suppose to the contrary that $ab + cd$ is prime. Note that

$$ab + cd = (a+d) + (b-c)a = m \cdot \gcd(a+d, b-c)$$

for some positive integer m. By assumption, either $m = 1$ or $\gcd(a+d, b-c) = 1$.

Case (i): $m = 1$. Then

$$\begin{aligned}\gcd(a+d, b-c) = ab + cd &> ab + cd - (a-b+c+d)\\ &= (a+d)(c-1) + (b-c)(a+1)\\ &\geq \gcd(a+d, b-c).\end{aligned}$$

which is false.

Case (ii): $\gcd(a+d), b-c) = 1$. Substituting $ac+bd = (a+d)b-(b-c)a$ for the left hand side of $a+c+bd = (b+d+a-c)(b+d-a+c)$, we obtain

$$(a+d)(a-c-d) = (b-c)(b+c+d).$$

In view of this, there exists a positive integer k such that

$$a-c-d = k(b-c),$$
$$b+c+d = k(a+d).$$

Adding we get $a+b = k(a+b-c+d)$ and thus $k(c-d) = (k-1)(a+b)$. Recall that $a > b > c > d$. If $k = 1$ then $c = d$, a contradiction. If $k \geq 2$ then

$$2 \geq \frac{k}{k-1} = \frac{a+b}{c-d} > 2,$$

a contradiction.

Third soln. The equality $ac+bd = (b+d+a-c)(b+d-a+c)$ is equivalent

$$a^2 - ac + c^2 = b^2 + bd + d^2. \tag{1}$$

Let $ABCD$ be the quadrilateral with $AB = a$, $BC = d$, $CD = b$, $AD = c$, $\angle BAD = 60°$ and $\angle BCD = 120°$. such a quadrilateral exists in view of (1) and the law of cosines; the common value in (1) is BD^2. Let $\angle ABC = \alpha$ so that $\angle CDA = 180° - \alpha$. Apply the law of cosines to $\triangle ABC$ and $\triangle ACD$ gives

$$a^2 + d^2 - 2ad\cos\alpha = AC^2 = b^2 + c^2 + 2bc\cos\alpha.$$

Hence $2\cos\alpha = (a^2+d^2-b^2-c^2)/(ad+bc)$, and

$$AC^2 = a^2 + d^2 - ad\frac{a^2+d^2-b^2-c^2}{ad+bc} = \frac{(ab+cd)(ac+bd)}{ad+bc}.$$

Because $ABCD$ is cyclic, Ptolemy's theorem gives

$$(AC \cdot BD)^2 = (ab+cd)^2.$$

It follows that

$$(ac+bd)(a^2-ac+c^2) = (ab+cd)(ad+bc). \tag{2}$$

Next observe that

$$ab+cd > ac+bd > ad+bc \tag{3}$$

The first follows from $(a-d)(b-c) > 0$ and the second from $(a-b)(c-d) > 0$.

Now assume that $ab+cd$ is prime. It then follows from (3) that $ab+cd$ and $ac+bd$ are relatively prime. Hence from (2), it must be true that $ac+bd$ divides $ad+bc$. However, this is impossible by (3). Thus $ab+cd$ is not prime.

Fourth soln. Consider the substitution: $w = -a+b+c+d, x = a-b+c+d, y = a+b-c+d, z = a+b+c-d$. Notice that $0 < w < x < y < z$ because $w > 0$ from the condition for $ac+bd$ and the other inequalities follow from $a > b > c > d$.This substitution gives: $a = (-w+x+y+z)/4$ etc Plug this into the condition for $ac+bd$ and we obtain: $3wy = xz$.

We want to show that $ab+cd = (wx+yz)/4$ is not prime. To do this, consider writing $w = 2^{a_1}3^{b_1}t_1$ where t_1 is a product of odd prime powers bigger than 3. Now, write similar expressions for x, y, z, using the subscripts $2, 3, 4$.

Suppose a prime $p > 3$ divides w, then because $3wy = xz$, $p \mid x$ or z. But if p divides z, then p divides $wx+yz$ and we are done, so assume p divides x. This implies $t_1 \mid t_2$. We can use a similar argument to show that $t_2 \mid t_1$. Hence $t_1 = t_2$. Similarly $t_3 = t_4$.

From $3wy = xz$, we have $a_1 + a_3 = a_2 + a_4$ and $1 + b_1 + b_3 = b_2 + b_4$. If $a_1 = a_2$, then from the inequality $0 < w < x < y < z$, we have $b_2 \geq b_1 + 1$, $b_3 \geq b_4$ and finally $a_3 < a_4$, a contradiction. Thus $a_1 \neq a_2$. Similarly, $a_3 \neq a_4$. Thus $a_1 + a_2 > 0$ and $a_3 + a_4 > 0$. Since $wx + yz = 2^{a_1+a_2}3^{b_1+b_2}t_1^2 + 2^{a_3+a_4}3^{b_3+b_4}t_3^2$. Hence $8 \mid wx+yz$ as $3^{b_1+b_2}t_1^2 + 3^{b_3+b_4}t_3^2 \equiv 4 \pmod 8$ since $b_1 + b_2$ and $b_3 + b_4$ have opposite parity.