43rd International Mathematical Olympiad

Glasgow, United Kingdom, July 2002

1. Let n be a positive integer. let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x + y < n$. Each point of T is coloured red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \leq x$ and $y' \leq y$. Define an X-set to be a set of n blue points having distinct x-coordinates, and a Y-set to be a set of n blue points have distinct y-coordinates. Prove that the number of X-sets is equal to the number of Y-sets.

Soln. Induct on the number B of blue points in the diagram. The base cases: If there are no blue points in the bottom row, then there are no X-sets and no Y-sets and so we are done.

The recursive case: If there is only one blue point at the bottom row, then delete the botton row to get a new configuration. The number of X-sets in the two configurations are equal as are the Y-sets. Thus we are done. If there are $m(\geq 2)$ blue points in the bottom row, then change the leftmost blue point P to red. In the new configuration, the number of X-sets and Y-sets are equal. Now restore the colour of P to red. The now of X-sets is changed by a factor of $m/(m-1)$. The same goes for the number of Y-sets. Thus are are done again.

Second soln. Assign the value $k/(k-1)$ to any blue point (x, y) with $x + y = n - k$, $k = 2, \ldots, n$. Any point (x, y) with $x + y = n - 1$ is assigned the value 0 if it is coloured red and is assigned the value 1 if it is coloured blue. The other red points are assigned the value 1. Then the number of blue points in each row or column is the product of the numbers in the row or column. Hence the number of X-sets and the number of Y-sets are both equal to the product of all the numbers assigned and they are therefore equal in number.

Third soln. First we observe that when all the points are red, the number of X- and Y-sets are both zero and so they are equal. Now suppose that in a given configuration, the number of X-and Y-sets are equal. If (x, y) is a red point with $x + y = n - k$ and all points (x, y'), (x', y) where $x' > x, y' > y$ are blue, then changing the colour of this point to blue changes the number of blue points in the row and column from $k - 1$ to k. Thus the number of X- and Y-sets remain equal in the new configuration. Since we can construct any configuration starting from the initial configuration with all points red by changing the red points to blue row by row, we conclude that the number of X- and Y-sets are equal in any configuration. (Taking this appraoch you must show that you can reach the configuration you want starting from the initial configuration, i.e., you must indicate how you can reach the final configuration.)

Fourth soln. This is the reverse of the third solution. If (x, y) is a blue point with $x + y = n - k$ and all points (x, y'), (x', y) where $x' < x, y' < y$ are red, then changing the colour of this point to red changes the number of blue points in the row and column from k

to $k-1$. The the number of X- and Y-sets are equal in the original configuration if and only if they are equal in the new configuration. If we continue this process, eventually we reach the configuration in which all the points are red. In this final configuration the number of X and Y-sets are equal. Therefore they are also equal in the original configuration. (Note: The advantage of this apprach over the previous one is that you need not worry about reaching the final configuration.)

2. Let BC be a diameter of the circle Γ with centre O. Let A be a point on Γ such that $0° < \angle AOB < 120°$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incentre of the triangle CEF.

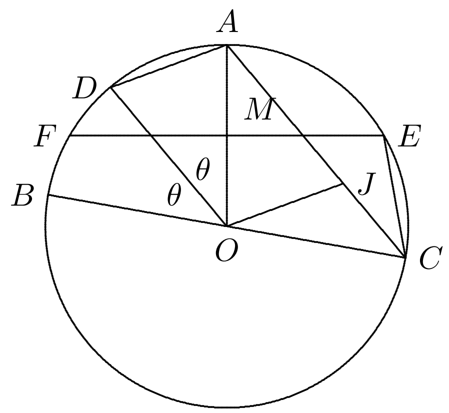
Soln. Let O be the origin and OA be the y-axis. We may also assume that the radius of Γ is 2. Let $\angle BOA = 2\theta$, then the coordinates of A, B, C, D, E, F are:

$$A(0,2),\ B(-2\sin 2\theta, 2\cos 2\theta),\ C(2\sin 2\theta, -2\cos 2\theta),$$
$$D(-2\sin\theta, 2\cos\theta),\ E(\sqrt{3},1),\ F(-\sqrt{3},1).$$

Then $AE = AF = 2$. Since arc AF = arc AE, AC bisects $\angle FCE$. (Note that the condition $\angle AOB < 120°$ implies that A and C are on the opposite sides of EF and so AC is the internal angle bisector.) Thus J is the incentre of ΔCEF if $AJ = AE = AF$. Thus we only need to show that $AJ = 2$. For some t,

$$\begin{aligned}
\overline{OJ} &= t\overline{DA} = t(2\sin\theta, 2-2\cos\theta)\\
\overline{AJ} &= (2t\sin\theta, 2t-2t\cos\theta-2)\\
\overline{CA} &= (2\sin 2\theta, -2\cos 2\theta - 2) = 2\cos\theta(2\sin\theta, -2\cos\theta)
\end{aligned}$$

Since CA is parallel to JA, we have $t = 1$ or $AJ = 2$ as required.



Second soln. The fact $0° < \angle AOB < 120°$ implies that $\angle AOC > \angle OAD$ and $\angle AOJ > \angle OAJ$. Thus J is an interior of the segment AC and A, J are on opposite sides of FE so the diagram is correct. A is the midpoint of arc EAF, so CA bisects $\angle ECF$. Now since $OA = OC$, $\angle AOD = \frac{1}{2}\angle AOB = \angle OAC$ so OD is parallel to JA and $ODAJ$ is a parallelogram. Hence $AJ = OD = OE = AF$ since $OEAF$ (with diagonals bisecting each other at right angles) is a rhombus. Thus

$$\begin{aligned} \angle JFE &= \angle JFA - \angle EFA = \angle AJF - \angle ECA \\ &= \angle AJF - \angle JCF = \angle JFC. \end{aligned}$$

Therefore, JF bisects $\angle EFC$ and J is the incentre of $\triangle CEF$.

3. Find all pairs of integers $m, n \geq 3$ such that there exist infinitely many positive integers a for which

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is an integer.

Soln. The problem is equivalent to the following: Find all pairs of integers $m, n \geq 3$ for which $x^n + x^2 - 1$ is a factor of the polynomial $x^m + x - 1$. It's clear that $m > n$. Write $m = n + k$. Then

$$x^m + x - 1 = x^k(x^n + x^2 - 1) + (1 - x)(x^{k+1} + x^k - 1).$$

So $x^n + x^2 - 1$ divides $x^{k+1} + x^k - 1$. Now $x^n + x^2 - 1$ has a real root $\alpha \in (0, 1)$, α is also a root of $x^{k+1} + x^k - 1$. Thus $\alpha^{k+1} + \alpha^k = 1$ and $\alpha^n + \alpha^2 = 1$. But $k + 1 \geq n \geq 3$ and so $\alpha^n + \alpha^2 \geq \alpha^{k+1} + \alpha^k$ with equality if and only if $k + 1 = n$ and $k = 2$. Thus $(m, n) = (5, 3)$ is the only possible solution. It is easy to check that this is indeed a solution.

4. Let n be an integer greater than 1. The positive divisors of n are $d_1, d_2, \ldots, d_k$, where

$$1 = d_1 < d_2 < \cdots < d_k = n.$$

Define $D = d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$.

(a) Prove that $D < n^2$.

(b) Determine all n for which D is a divisor of n^2.

Soln. If d is a divisor of n, then so is n/d and $n/d_k < \cdots < n/d_2 < n/d_1$. Thus

$$D = n^2 \sum \frac{1}{d_i d_{i+1}} \leq n^2 \sum \left(\frac{1}{d_i} - \frac{1}{d_{i+1}}\right) < \frac{n^2}{d_1} = n^2.$$

For part (b), note that if n is a prime, then $D = n$ which certainly divides n^2. Now suppose that n is not prime. As d_{k-1} is the greatest proper divisor of n, then $d_{k-1}d_k = d_{k-1}n$ is the greatest proper divisor of n^2. But $n^2 > D > d_{k-1}d_k$. So D is not a divisor of n^2. Therefore D is a divisor of n^2 if and only if n is prime.

5. Find all functions f from the set $\mathbb{R}$ of real numbers to itself such that

$$\big(f(x)+f(z)\big)\big(f(y)+f(t)\big) = f(xy-zt)+f(xt+yz) \qquad (*)$$

for all $x, y, z, t \in \mathbb{R}$.

Soln. It is clear that $f(x) \equiv 0$, $f(x) \equiv 1/2$ and $f(x) = x^2$ for all x are solutions. We claim that there are no other solutions.

Setting $x = y = z = 0$ gives $2f(0) = 2f(0)\big(f(0)+f(t)\big)$. In particular $2f(0) = 4f(0)^2$ and so $f(0) = 0$ or $1/2$. If $f(0) = 1/2$ we get $f(0)+f(t) = 1$ and so $f(x) \equiv 1/2$.

Suppose $f(0) = 0$. Then setting $z = t = 0$ in $(*)$ gives $f(xy) = f(x)f(y)$. In particular $f(1) = f(1)^2$ and so $f(1) = 0$ or 1. If $f(1) = 0$, then $f(x) = f(x)f(1) = 0$ for all x.

So we may assume that $f(0) = 0$ and $f(1) = 1$. Setting $x = 0$ and $y = t = 1$ in $(*)$, we have

$$f(-z)+f(z) = 2f(z) \quad \text{or} \quad f(-z) = f(z)$$

and f is an even function. So it suffices to show that $f(x) = x^2$ for positive x. Note that $f(x^2) = f(x)^2 \geq 0$; as f is even, we have $f(y) \geq 0$ for all y. Now put $t = x$ and $z = y$ in $(*)$ to get

$$f(x^2+y^2) = \big(f(x)+f(y)\big)^2.$$

This shows that $f(x^2+y^2) \geq f(x)^2 = f(x^2)$. Hence f is increasing on the positive reals. Set $y = z = t = 1$ in $(*)$ to yield

$$f(x-1)+f(x+1) = 2(f(x)+1).$$

By induction on n, it readily follows that $f(n) = n^2$ for all non-negative integers n. As f is even, $f(n) = n^2$ for all integers n. Since f is multiplicative, $f(r) = r^2$ for all rational numbers r. Suppose $f(x) \neq x^2$ for some positive x. If $f(x) < x^2$, take a rational number a with $x > a > \sqrt{f(x)}$. Then $f(a) = a^2 > f(x)$, but $f(a) \leq f(x)$ as f is increasing on positive reals. This is a contradiction. A similar argument shows that $f(x) > x^2$ is impossible. Thus $f(x) = x^2$ for all real x.

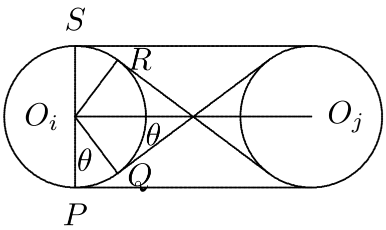
6. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be circles of radius 1 in the plane, where $n \geq 3$. Denote their centres by $O_1, O_2, \ldots, O_n$, respectively. Suppose that no line meets more than two of the circles. Prove that

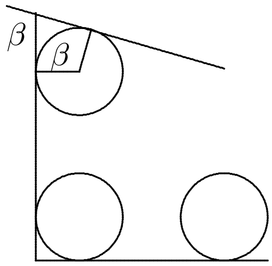
$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}.$$

Soln. Consider the circles Γ_i, Γ_j and their 4 common tangents. The circle Γ_i contains 2 minor arcs PQ, RS, each of length

$$\theta_{ij} \geq \sin\theta_{ij} = \frac{2}{O_iO_j}.$$

The tangent at any interior point of these minor arcs will intersect only Γ_j. Thus for each fixed i, the minor arcs obtained as j varies are disjoint.





Now enclose the n circles by a convex polygon so that each side is tangent to at least 2 of the circles. So the two sides at vertex V_k is tangent to a circle Γ. The two points of contact define a minor arc. The length of the minor arc is eqaul to the external angle β_k at V_k and $\sum_k \beta_k = 2\pi$. These minor arcs are disjoint from the minor arcs described earlier. Thus

$$\sum_i \sum_j 2\theta_{ij} + \sum_k \beta_k \leq 2n\pi \quad \text{and} \quad \sum_i \sum_j 2\theta_{ij} = 2(n-1)\pi.$$

Therefore

$$2(n-1)\pi \geq \sum_i \sum_j 2\theta_{ij} = \sum_{1 \leq i < j \leq n} 4\theta_{ij} \geq \sum_{1 \leq i < j \leq n} \frac{8}{O_i O_j}$$

as required.