35th International Mathematical Olympiad

Hong Kong, July 1994.

1. Let m and n be positive integers. let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \le n$ for some i, j, $1 \le i \le j \le m$, there exists k, $1 \le k \le m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \ge \frac{n+1}{2}.$$

Soln. Without loss of generality, we may assume that $a_1 > a_2 > \cdots > a_m$. We claim that $a_i + a_{m+1-i} \ge n + 1$ for $i = 1, \ldots, m$. The result then follows readily. To prove the claim, we assume that on the contrary that it's false. Thus there exists i such that $a_i + a_{m+1-i} < n + 1$ Then $a_i < a_i + a_m < a_i + a_{m-1} < \cdots < a_i + a_{m+1-i} \le n$. Thus

$$\{a_i + a_m, a_i + a_{m-1}, \ldots, a_i + a_{m+1-i}\} \subseteq \{a_1, a_2, \ldots, a_{i-1}\}$$

which is impossible. Thus the claim follows.

2. ABC is an isosceles triangle with $AB = AC$. Suppose that

(i) M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;

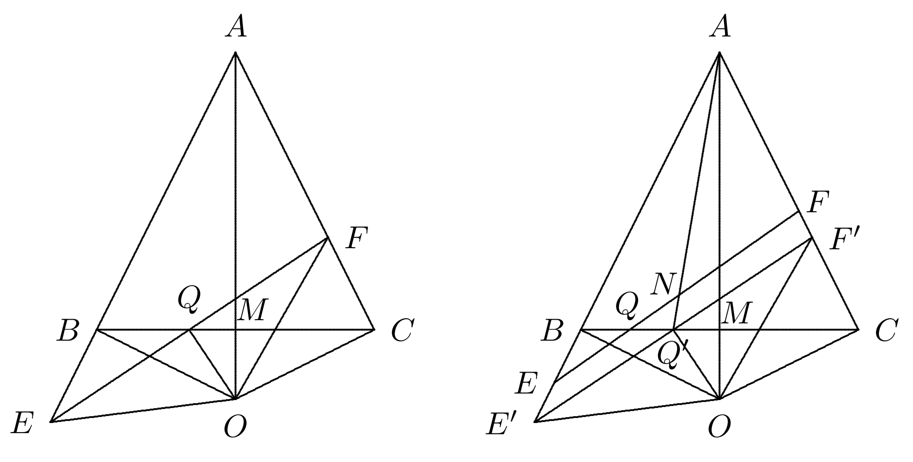
(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F lies on the line AC such that E, Q and F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

Soln. First assume that OQ is perpendicular to EF. Now $OEBQ$ abd $OCFQ$ are cyclic. Hence $\angle OEQ = \angle OBQ = \angle OCQ = \angle OFQ$. It follows that $QE = QF$.

Suppose now that $QE = QF$ and that the perpendicular through O to EF meet BC at $Q' \ne Q$. Draw the line through Q' parallel to EF, meeting the lines AB and AC at E' and F', respectively. Then $Q'E' = Q'F'$ as before. Let AQ' meet EF at N. Then $N \ne Q$ and $NE = NF$, so that $QE \ne QF$, a contradiction. So $Q' = Q$.



3. For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation has precisely three 1s.

(a) Prove that, for each positive integer m, there exists at least one positive integer k such that $f(k) = m$.

(b) Determine all positive integers m for which there exists exactly one k with $f(k) = m$.

Soln. Let $g(k)$ denote the number of elements in the set $\{1, \ldots, n\}$ whose binary representation has exactly three ones. Then $f(k)$ and $g(k)$ are both increasing and $f(k) = g(2k) - g(k)$. Hence

$$\begin{aligned} f(k+1) - f(k) &= g(2k+2) - g(k+1) - g(2k) + g(k) \\ &= g(2k+2) - g(2k) - [g(k+1) - g(k)] \end{aligned}$$

Since either both $2k+2$ is counted in $g(2k+2)$ and $k+1$ is counted in $g(k+1)$ or neither is. Thus $f(k+1) - f(k)$ is either 1 or 0 depending on where $2k+1$ is counted in $g(2k+1)$ or not. Since $f(2^n) = \binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$, the image of f is $\mathbb{N} \cup \{0\}$. This proves (a).

Let m be any positive integer for which there is only one k with $f(k) = m$. Then

$$f(k+1) - f(k) = 1 = f(k) - f(k-1).$$

The former means $2k+1$ is counted in $g(2k+2)$, or equivalently, the binary representation of k has exactly two ones. The same holds for $k-1$. This happens only when the last two digits of $k-1$ are 01. In other words, $k = 2^n + 2$. But

$$\begin{aligned} f(2^n + 2) &= g(2^{n+1} + 4) - g(2^n + 2) \\ &= 1 + g(2^{n+1} - g(2^n) \\ &= 1 + \binom{n}{2} \end{aligned}$$

Thus the answer is any number of the form $1 + \binom{n}{2}$, $n \geq 2$.

4. Determine all ordered pairs (m, n) of positive integers such that

$$\frac{n^3 + 1}{mn - 1}$$

is an integer.

Soln. Note that $mn - 1$ and m^3 are relatively prime. That $mn - 1$ dividing $n^3 + 1$ is therefore equivalent to $mn - 1$ dividing $m^3(n^3 + 1) = m^3n^3 - 1 + m^3 + 1$, which is in turn equivalent to $mn - 1$ dividing $m^3 + 1$. If $m = n$, we have $\frac{n^3+1}{n^2-1} = n + \frac{1}{n-1}$. This is an integer if and only if $n = 2$. We now consider the case $m > n$. If $n = 1$, $\frac{2}{m-1}$ is an integer. This is so if and ly if $m = 2, 3$. Suppose $n \geq 2$. Note that $n^3 + 1 \equiv 1 \pmod{n}$ while $mn - 1 \equiv -1$

(mod n). Hence $\frac{n^3+1}{mn-1} = kn-1$ for some positive integer k. Now $kn-1 < \frac{n^3+1}{n^2-1} = n + \frac{1}{n-1}$ or $(k-1)n < 1 + \frac{1}{n-1}$. Hence $k = 1$, so that $n^3+1 = (mn-1)(n-1)$. This yields $m = \frac{n^2+1}{n-1} = n+1+\frac{2}{n-1}$, which is an integer if and only if $n = 2, 3$. In each case, we have $m = 5$. In summary, there are 9 solutions, namely

$$(2,2),(2,1),(3,1),(5,2),(5,3),(1,2),(1,3),(2,5),(3,5)$$

the last 4 obtained by symmetry.

5. Let $\mathbb{S}$ be the set of real numbers strictly greater than -1. Find all functions $f : \mathbb{S} \to \mathbb{S}$ satisfying the two conditions:

(i) $f(x) + f(y) + xf(y) = y + f(x) + yf(x)$ for all x and y in $\mathbb{S}$;

(ii) $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

Soln. Conditionm (ii) implies that $f(x) = x$ has at most three solutions, one in $(-1, 0)$, one equal to 0 and the third in $(0, \infty)$.

Suppose $f(u) = u$ for some $u \in (-1, 0)$. Setting $x = y = u$ in (i), we get

$$f(u^2 + 2u) = u^2 + 2u \in (-1, 0).$$

This means $u^2 + 2u = u$. But then $u \notin (-1, 0)$. The case $f(v) = v$ for some $v > 0$ leads to a similar contradiction.

However, $f(x+(1+x)f(x)) = x+(1+x)f(x)$ for all $x \in \mathbb{S}$. So we have $x+(1+x)f(x) = 0$ which gives $f(x) = -\frac{x}{1+x}$.

It's routine to check that $f(x) = -\frac{x}{1+x}$ satisfies the desired property.

6. Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \in A$ and $n \notin A$ each of which is a product of k distinct elements of S for some $k \geq 2$.

Soln. Let A be the of all positive integers of the form $q_1 q_2 \ldots q_{q_1}$ where $q_1 < q_2 < \cdots < q_{q_1}$ are primes. For any infinite set $S = \{p_1, p_2, p_3, \ldots\}$ of primes with $p_1 < p_2 < \cdots$, we can satisfy he requirement of the problem by taking $k = p_1$, $m = p_1 p_2 \cdots p_k$ and $n = p_2 p_3 \cdots p_{k+1}$.