Singapore International Mathematical Olympiad Junior Group Test

Take Home Test - Number Theory **13-8-2005**

1. Suppose a, b are integers satisfying $24a^2 + 1 = b^2$. Prove that exactly one of a, b is divisible by 5.

2. Let $x = \overline{abcd}$ be a 4-digit number such that the last 4 digits of x^2 are also $\overline{abcd}$. Find all possible values of x.

3. Prove that for any positive integer n, $19 \mid 5^{2n+1} + 3^{n+2}2^{n-1}$.

4. Let a, b, c be odd integers. Consider the equation

$$ax^2 + bx + c = 0.$$

Prove that $x = p/q$ where p, q are integers cannot be a solution.

5. Solve the following equation in integers:

$$4x^2 + 4x = y^2 + y.$$

Take Home Test - Geometry

6. In figure A, $DE = FB$, $\angle ADE = \angle EDC$, $\angle EBF = \angle FBC$, and $\angle DAB = \angle BCD$. Prove that $AE = BC$.

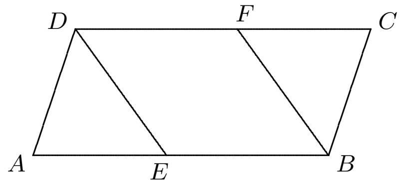


figure A.

7. In the convex quadrilateral $ABCD$, $AB = BC$, E is a point inside the quadrilateral such that $BCDE$ is a parallelogram. Let P, Q, R and S be the midpoints of the segments BD, DA, AE and EB respectively. Prove that $PQRS$ is a rhombus.

8. In $\triangle ABC$, M and N are points on AB and AC respectively such that $BM = CN$. Let D and E be the midpoints of MN and BC respectively. Prove that the line through A parallel to DE bisects $\angle A$.

9. Let M and N be points respectively on the sides AB and CD of the quadrilateral $ABCD$, and let E and F be the feet of the perpendiculars from M onto CD and from N onto AB respectively. Suppose $CN = BM$, $ND = MA$ and $ME = NF$. Prove that AD is parallel to BC.

10. Let $A_1A_2A_3$ be a triangle and let B_1, B_2, B_3 be points on sides A_2A_3, A_3A_1, A_1A_2 respectively (not coinciding with any vertices). Prove that the perpendicular bisectors of the three segments A_1B_1, A_2B_2, A_3B_3 do not pass through a common point.

Due on 20-8-2004

Solution to Take Home Test - Number Theory **13-8-2005**

1. Suppose a, b are integers satisfying $24a^2+1 = b^2$. Prove that exactly one of a, b is divisible by 5.

   Solution. Taking modulo 5, we get $-a^2 + 1 \equiv b^2 \pmod 5$ or $a^2 + b^2 \equiv 1 \pmod 5$. Since $a^2 \equiv 0, 1, 4 \pmod 5$, we see that the only possibility is that one of a^2, b^2 is 0 and the other 1 $\pmod 5$.

2. Let $x = \overline{abcd}$ be a 4-digit number such that the last 4 digits of x^2 are also $\overline{abcd}$. Find all possible values of x.

   Solution. We have $10000 \mid x^2 - x = x(x-1)$. Since x and $x-1$ are coprime, and $10000 = 2^4 5^4$, we have either $16 \mid x$ and $625 \mid x-1$ or $16 \mid x-1$ and $625 \mid x$.

   The 4-digit odd multiples of 625 are:

   $$1875, 3135, 4375, 5625, 6875, 8125, 9375.$$

   If 1 is added, only $16 \mid 9375 + 1$. If 1 is subtracted, then none is divisible by 16. So $x = 9376$ is the only answer.

3. Prove that for any positive integer n, $19 \mid 5^{2n+1} + 3^{n+2}2^{n-1}$.

   Solution 1. $5^{2n+1} + 3^{n+2}2^{n-1} = 125 \times 25^{n-1} + 27 \times 6^{n-1} \equiv 6^{n-1}(11 + 8) \equiv 0 \pmod{19}$.

   Solution 2. Let $a_n = 5^{2n+1} + 3^{n+2}2^{n-1}$. Then $a_1 = 152 \equiv 0 \pmod{19}$. Also

   $$a_{n+1} \equiv 5^{2n+3}+3^{n+3}2^n \equiv 25(5^{2n+1})+6(3^{n+2}2^{n-1}) \equiv 6(5^{2n+1}+3^{n+2}2^{n-1}) \equiv 6a_n \pmod{19}$$

   Thus $a_{n+1} \equiv 6^n a_1 \equiv 0 \pmod{19}$

4. Let a, b, c be odd integers. Consider the equation

   $$ax^2 + bx + c = 0.$$

   Prove that $x = p/q$ where p, q are integers cannot be a solution.

   Solution 1. If $x = p/q$ satisfies the equation, then

   $$ap^2 + bpq + cq^2 = 0.$$

   We may assume that p, q are coprime. Thus p, q cannot be both even.

   If p, q both odd, the lefthand side is odd.

   If p, q are of opposite parity, then the lefthand side is also odd.

   In both cases, we get a contradiction. Thus no such solution exists.

   Solution 2. $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Thus for $x = p/q$ to be a solution, $b^2 - 4ac = m^2$ for some integer m. Since b is odd, so m is odd as well. Thus $b^2 \equiv m^2 \equiv 1 \pmod 8$. Thus we have $4ac \equiv 0 \pmod 8$. But this means ac is even, a contradiction. Thus no such solution exists.

5. Solve the following equation in integers:

   $$4x^2 + 4x = y^2 + y.$$

   Solution 1. When $y = -1$, we get $x = 0, -1$. When $y = 0$, we get $x = 0, -1$.

   When $y > 0$, adding 1 to both sides, we get $(2x + 1)^2 = y^2 + y + 1$. But $y^2 < y^2 + y + 1 < y^2 + 2y + 1 = (y + 1)^2$ Thus $y^2 + y + 1$ is not a square. Hence there are no solutions.

   When $y < -1$,adding 1 to both sides, we get $(2x+1)^2 = y^2+y+1$. But $y^2 > y^2+y+1 > y^2 + 2y + 1 = (y + 1)^2$. Again $y^2 + y + 1$ is not square and so there no solutions.

Solution 2. Since $4x^2 + 4x - (y^2 + y) = 0$, we have $x = \frac{-4 \pm \sqrt{16+16(y^2+y)}}{8} = \frac{-1 \pm \sqrt{y^2+y+1}}{2}$. For x to be an integer, we need $y^2 + y + 1$ to be a square. It is a square when $y = 0$, which yields $x = 0, -1$ or when $y = -1$ which also gives $x = 0, -1$. For other values of y, you can prove that $y^2 + y + 1$ is not a square as in solution 1.

Solution to Take Home Test - Geometry

6. In figure A, $DE = FB$, $\angle ADE = \angle EDC$, $\angle EBF = \angle FBC$, and $\angle DAB = \angle BCD$. Prove that $AE = BC$.

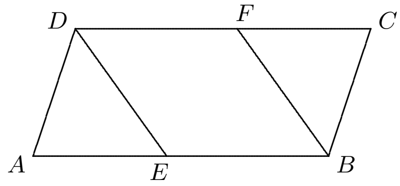
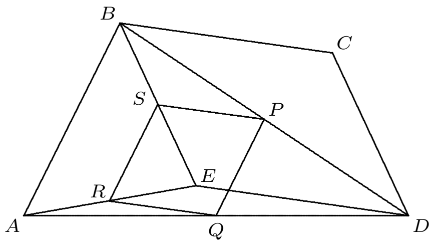


figure A.

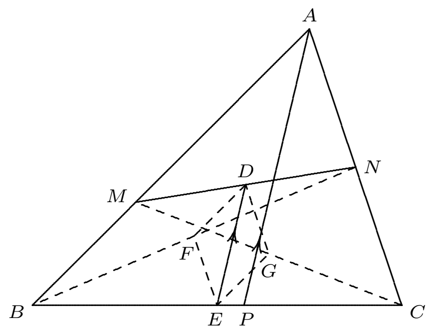
Solution. Produce DE and CB meeting at a point G. Then $\angle BGE = \angle AED = \angle BEG$. Thus $\angle BEG = \angle ABF$ so that $\angle AED = \angle CBF$. Note that $DE \parallel FB$ and $DE = FB$ which mean $DFBE$ is a parallelogram. Then $\triangle ADE$ and $\triangle CBF$ are congruent isosceles triangles.

7. In the convex quadrilateral $ABCD$, $AB = BC$, E is a point inside the quadrilateral such that $BCDE$ is a parallelogram. Let P, Q, R and S be the midpoints of the segments BD, DA, AE and EB respectively. Prove that $PQRS$ is a rhombus.



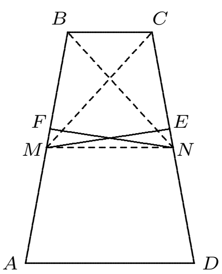
Solution. As $RS = \frac{1}{2}AB = PQ$, $PS = \frac{1}{2}ED = RQ$ and $PS = \frac{1}{2}ED = \frac{1}{2}BC = \frac{1}{2}AB = RS$, we have $RS = PS = PQ = RQ$. Thus $PQRS$ is a rhombus.

8. In $\triangle ABC$, M and N are points on AB and AC respectively such that $BM = CN$. Let D and E be the midpoints of MN and BC respectively. Prove that the line through A parallel to DE bisects $\angle A$.



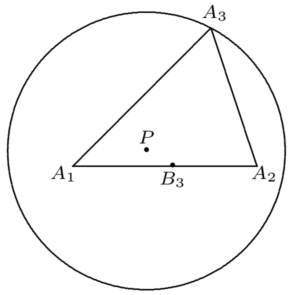
Solution. Let F and G be the midpoints of BN and MC respectively. Then $DFEG$ is a parallelogram. Furthermore, $DF = \frac{1}{2}MB = \frac{1}{2}NC = DG$. Similarly, $EF = GD$. Thus $DFEG$ is a rhombus. Therefore, $\angle BAP = \angle FDE = \angle GDE = \angle CAP$.

9. Let M and N be points respectively on the sides AB and CD of the quadrilateral $ABCD$, and let E and F be the feet of the perpendiculars from M onto CD and from N onto AB respectively. Suppose $CN = BM$, $ND = MA$ and $ME = NF$. Prove that AD is parallel to BC.



Solution. Join MN, MC and NB. Then, $S_{\triangle CMN} = \frac{1}{2}CN \cdot ME = \frac{1}{2}NF \cdot BM = S_{\triangle BMN}$. Therefore, the distance from B to MN is equal to the distance from C to MN. Hence, BC is parallel to MN. Similarly, AD is parallel to MN.

10. Let $A_1A_2A_3$ be a triangle and let B_1, B_2, B_3 be points on sides A_2A_3, A_3A_1, A_1A_2 respectively (not coinciding with any vertices). Prove that the perpendicular bisectors of the three segments A_1B_1, A_2B_2, A_3B_3 do not pass through a common point.



Solution. Suppose they concur at P. Assume without loss of generality that $PA_1 \le PA_2 \le PA_3$. Then A_1, A_2 lie in the closed disc with centre P and radius PA_3, hence B_3 lies inside that disc, and we have $PB_3 < PA_3$. On the other hand, it has been assumed that P lies on the perpendicular bisector of A_3B_3, and so $PA_3 = PB_3$ which is a contradiction.