Singapore International Mathematical Olympiad
National Team Selection Test I 2006

Time allowed: 4.5 hours **12 May 2006**

No calculator is allowed

1. Let ANC, CLB and BKA be triangles erected on the outside of the triangle ABC such that $\angle NAC = \angle KBA = \angle LCB$ and $\angle NCA = \angle KAB = \angle LBC$. Let D, E, G and H be the midpoints of AB, LK, CA and NA respectively. Prove that $DEGH$ is a parallelogram.

2. Let n be an integer greater than 1 and let $x_1, x_2, \ldots, x_n$ be real numbers such that
$$|x_1| + |x_2| + \cdots + |x_n| = 1 \text{ and } x_1 + x_2 + \cdots + x_n = 0.$$
Prove that
$$\left|\frac{x_1}{1} + \frac{x_2}{2} + \cdots + \frac{x_n}{n}\right| \leq \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

3. A pile of n pebbles is placed in a vertical column. This configuration is modified according to the following rules. A pebble can be moved if it is at the top of a column which contains at least two more pebbles than the column immediately to its right. (If there are no pebbles to the right, think of this as a column with 0 pebbles.) At each stage, choose a pebble from among those that can be moved (if there are any) and place it at the top of the column to its right. If no pebbles can be moved, the configuration is called a *final configuration*. For each n, show that, no matter what choices are made at each stage, the final configuration obtained is unique. Describe that configuration in terms of n.

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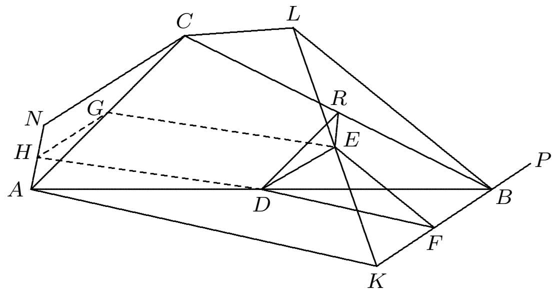
1. In the plane containing a triangle ABC, points A', B' and C' distinct from the vertices of $\triangle ABC$ lie on the lines BC, AC and AB respectively such that AA', BB' and CC' are concurrent at G and $AG/GA' = BG/GB' = CG/GC'$. Prove that G is the centroid of $\triangle ABC$.

2. Let S be a set of sequences of length 15 formed by using the letters a and b such that every pair of sequences in S differ in at least 3 places. What is the maximum number of sequences in S?

3. Let n be a positive integer such that the sum of all its positive divisors (inclusive of n) equals to $2n + 1$. Prove that n is an odd perfect square.

1. Let ANC, CLB and BKA be triangles erected on the outside of the triangle ABC such that $\angle NAC = \angle KBA = \angle LCB$ and $\angle NCA = \angle KAB = \angle LBC$. Let D, E, G and H be the midpoints of AB, LK, CA and NA respectively. Prove that $DEGH$ is a parallelogram.

Solution. The given condition means that $\triangle ANC$, $\triangle CLB$ and $\triangle BKA$ are all similar.



Let F be the midpoint of BK. Note that DF is parallel to AK and EF is parallel LB. Since $\angle DFE = 180° - \angle DFK - \angle EFB = 180° - (\angle KAB + \angle ABK) - \angle LBP = 180° - \angle LBC - \angle ABK - \angle LBP = \angle ABC$ and $DF/EF = AK/LB = AB/CB$, we see that $\triangle DFE$ is similar to $\triangle ABC$. Thus $\angle BAC - \angle ACN = \angle FDE - \angle FDB$ so that DE is parallel to NC and HG. Next, we also have $\angle EDR = \angle FDR - \angle FDE = \angle FDB + \angle BDR - \angle FDE = \angle KAB + \angle BAC - \angle FDE = \angle KAB$, and $DE/DR = 2DE/AC = 2DF/AB = AK/AB$ so that $\triangle EDR$ is similar to $\triangle KAB$. That means $\triangle EDR$ is similar to $\triangle NCA$. Therefore, $DE/HG = 2DE/NC = 2DR/AC = 1$. Consequently, DE is parallel and equal to HG. This shows that $DEGH$ is a parallelogram.

[Remark by Lim Wei Quan] Let M be the 4th vertice of the parallelogram $AKBM$. Let F be the midpoint of BC. Then since triangles BMD, BLF, CNG are similar, $ML = DF \times BL/BF = AC/2 \times 2CN/CA = CN$ Similarly, $MN = CL$ Thus, $MNCL$ is a parallelogram This gives, $DE = ML/2 = CN/2 = GH$ Also, $DE||ML||CN||GH$ Therefore, $DEGH$ is a parallelogram.

2. Let n be an integer greater than 1 and let $x_1, x_2, \ldots, x_n$ be real numbers such that

$$|x_1| + |x_2| + \cdots + |x_n| = 1 \quad \text{and} \quad x_1 + x_2 + \cdots + x_n = 0.$$

Prove that

$$\left|\frac{x_1}{1} + \frac{x_2}{2} + \cdots + \frac{x_n}{n}\right| \leq \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

Solution. The following lemma can be proved by direct simplification.

Lemma. Let $S_k = a_1 + a_2 + \cdots + a_k$. Then

$$\sum_{k=1}^{n} a_k b_k = S_n b_n + \sum_{k=1}^{n-1} S_k (b_k - b_{k+1}).$$

Let $S_i = x_1 + x_2 + \cdots + x_i$. By the given condition, $S_n = 0$ and $|S_i| \le \frac{1}{2}$ for $i = 1, \cdots, n-1$. To see this, suppose $|S_i| > \frac{1}{2}$. Then $1 = |x_1| + |x_2| + \cdots + |x_n| \ge |x_1 + \cdots + x_i| + |x_{i+1} + \cdots + x_n| = |S_i| + |-S_i| = 2|S_i| > 1$, which is a contradiction. By the lemma, we have

$$\sum_{k=1}^{n} \frac{x_k}{k} = S_n \cdot \frac{1}{n} + \sum_{k=1}^{n-1} S_k \left(\frac{1}{k} - \frac{1}{k+1}\right).$$

Thus

$$\left|\sum_{k=1}^{n} \frac{x_k}{k}\right| = \left|\sum_{k=1}^{n-1} S_k \left(\frac{1}{k} - \frac{1}{k+1}\right)\right| \le \sum_{k=1}^{n-1} |S_k| \left(\frac{1}{k} - \frac{1}{k+1}\right)$$
$$\le \sum_{k=1}^{n-1} \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

Second Solution. The inequality is achievable when $x_1 = \pm\frac{1}{2}$ and $x_n = \mp\frac{1}{2}$ and the rest of $x_i = 0$. So the inequality can be proved by the smoothing principle.

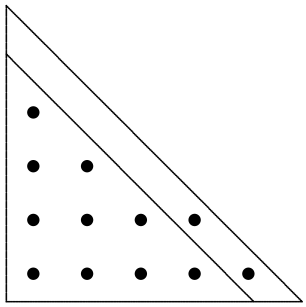
Let $a_1 \ge \cdots \ge a_k \ge 0$ be the nonnegative terms among the x_i's and $b_1 \le b_2 \le \cdots \le b_l < 0$ be the negative terms among the x_i's. Then we have $a_1 + \cdots + a_k = 1/2$ and $b_1 + \cdots + b_l = -1/2$.

Without loss of generality, we can assume that the contribution from the nonnegative terms are greater than the contributions from the negative terms in the LHS. Note that for $0 < i < j$, and $x, y \ge 0$, we have $x/i + y/j \le (x+y)/i + 0/j$. Applying this, we see that the LHS is less than or equal to

$$\frac{\sum_{i=1}^{k} a_i}{1} + \frac{0}{2} + \cdots + \frac{0}{n-1} + \frac{\sum_{i=1}^{k} b_i}{n} = \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

3. A pile of n pebbles is placed in a vertical column. This configuration is modified according to the following rules. A pebble can be moved if it is at the top of a column which contains at least two more pebbles than the column immediately to its right. (If there are no pebbles to the right, think of this as a column with 0 pebbles.) At each stage, choose a pebble from among those that can be moved (if there are any) and place it at the top of the column to its right. If no pebbles can be moved, the configuration is called a *final configuration*. For each n, show that, no matter what choices are made at each stage, the final configuration obtained is unique. Describe that configuration in terms of n.

Solution. At any stage, let p_i be the number of pebbles in column i for $i = 1, 2, \ldots$, where column 1 denotes the leftmost column. We will show that in the final configuration, for all i for which $p_i > 0$ we have $p_i = p_{i+1} + 1$, except that for at most one i^*, $p_{i^*} = p_{i^*+1}$. Therefore, the configuration looks like the figure shown below, where there are c nonempty columns and there are from 1 to c pebbles in the last diagonal row of the triangular configuration. In particular, let $t_k = 1 + 2 + \cdots + k = k(k+1)/2$ be the kth triangular number. Then c is the unique integer for which $t_{c-1} < n \le t_c$. Let $s = n - t_{c-1}$. Then there are s pebbles in the rightmost diagonal, and so the two columns with the same height are columns $c - s$ and $c - s + 1$ (except if $s = c$, in which case no nonempty columns have equal height).



Final Configuration for $n = 12$

Another way to say this is

$$p_i = \begin{cases} c - i & \text{if } i \le c - s, \\ c - i + 1 & \text{if } i > c - s. \end{cases} \tag{1}$$

To prove this claim, we show that

(a) At any stage of the process, $p_1 \ge p_2 \ge \cdots$.

(b) At any stage, it is not possible for there to be $i < j$ for which $p_i = p_{i+1}$, $p_j = p_{j+1}$, and $p_{i+1} - p_j \le j - i - 1$ (that is, the average decrease per column from column $i + 1$ to column j is 1 or less).

(c) At any final configuration, $p_i - p_{i+1} = 0$ or 1, with at most one i for which $p_i > 0$ and $p_i - p_{i+1} = 0$.

In the proofs of (a), (b) and (c), we use the following terminology. Let a *k-switch* be the movement of one pebble from column k to column $k+1$, and for any column i let a *drop* be the quantity $p_i - p_{i+1}$.

To prove (a), suppose a sequence of valid moves resulted in $p_i < p_{i+1}$ for the first time at some stage. Then the move leading to this stage must have been an *i-switch*, but it would be contrary to the condition that column i have at least 2 more pebbles than column $i + 1$, to allow switches.

To prove (b), if such a configuration were obtainable, there would be a minimum value of $j - i$ overall such obtainable configurations, and we now show that there is no minimum. Suppose $p_1, p_2, \ldots$ was such a minimal configuration. It cannot be that $j = i + 1$, for what would columns $i, i + 1, i + 2$ look like just before the move that made the height equal? The move must have been a *k-switch* for $i - 1 \le k \le i + 2$, but if so the configuration before the switch was impossible (not decreasing).

Now suppose $j > i+1$. Consider the first configuration C in the sequence for which columns $i, i+1, j, j+1$ are at their final heights. Note that from p_{i+1} to p_j the column decrease by exactly one each time in C, because if there was a drop of 2 or more at some point, there would have to be another drop of 0 in this interval to obtain an average of 1 or less, and thus $j - i$ is not minimal. The move leading to C was either an *i-switch* or a *j-switch*. If it was the former, at the previous stage columns $i+1$ and $i+2$ had the same height, violating the minimality of $j - i$. A similar contradiction arises if the move was a *j-switch*.

Finally, to prove (c), if any drop is 2 or more, the configuration isn't final. However, if all drops are 0 or 1, and there were two drops of 0 between nonempty columns (say between i and $i+1$ and between j and $j+1$), then (b) would be violated. Thus a final configuration that satisfied (b) also satisfies (c).

4. In the plane containing a triangle ABC, points A', B' and C' distinct from the vertices of $\triangle ABC$ lie on the lines BC, AC and AB respectively such that AA', BB' and CC' are concurrent at G and $AG/GA' = BG/GB' = CG/GC'$. Prove that G is the centroid of $\triangle ABC$.

Solution. We take all segments to be directed segments: thus $AG = -GA$, etc. We are given the condition

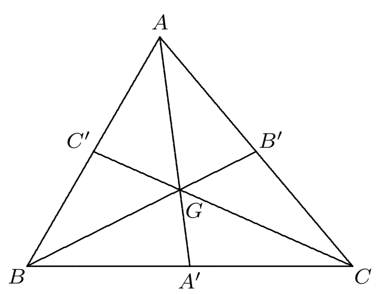
$$\frac{AG}{GA'} = \frac{BG}{GB'} = \frac{CG}{GC'}. \tag{1}$$

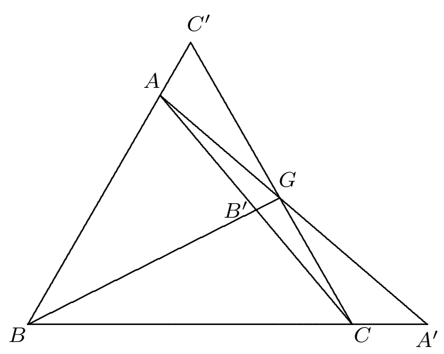
Now B, A' and C are collinear points, one on each side (extended if necessary) of triangle AGB'. By Menelaus' theorem, we have

$$\frac{AA'}{A'G} \cdot \frac{GB}{BB'} \cdot \frac{B'C}{CA} = -1. \tag{2}$$

Similarly, from triangle CGB' and collinear points C', B and A, we have

$$\frac{CC'}{C'G} \cdot \frac{GB}{BB'} \cdot \frac{B'A}{AC} = -1. \tag{3}$$





By adding 1 to each member of $AG/GA' = CG/GC'$ from (1), we have

$$\frac{AA'}{GA'} = \frac{CC'}{GC'}. \tag{4}$$

An easy combination of (2),(3),(4) gives

$$CB' = B'A. \tag{5}$$

Since these are directed segments, (5) means that B' is the midpoint of AC. Similarly, by selecting other triangles, we obtain C' as midpoint of AB, and A' as midpoint of BC. Thus G is the centroid of $\triangle ABC$.

So (1) implies that A', B', C' lie in the open segments BC, CA, AB, respectively, and the "open segments" restriction in the hypothesis is unnecessary. In order for any other conclusion to hold, we must interpret the symbols in (1) to mean undirected segments. Then, upon replacing them by directed segments we see that either all ratios have the same sign - the case already discussed - or two have one sign and the third the opposite sign. Suppose, say,

$$\frac{AG}{GA'} = \frac{CG}{GC'} = -\frac{BG}{GB'}. \tag{6}$$

By the original argument, the first equation implies (5) so that B' is the midpoint of AC. From $AG/GA' = -BG/GB'$, we have

$$\frac{AG}{GA'} + \frac{GA'}{GA'} - 1 = -\frac{BG}{GB'} - \frac{GB'}{GB'} + 1,$$

whence

$$\frac{AA'}{GA'} = 2 - \frac{BB'}{GB'}. \tag{7}$$

Now (5) implies that $B'C/CA = -1/2$, whence (2) becomes

$$\frac{AA'}{A'G} \cdot \frac{GB}{BB'} = 2. \tag{8}$$

Putting $GB/BB' = GB'/BB' - 1$ in (8) and eliminating $AA'/A'G$ from (7) and (8), we obtain

$$2r^2 - r + 1 = 0, \tag{9}$$

where $r = GB'/BB'$. But the roots of (9) are not real, so that this case cannot hold. Thus the case of medians is the only conclusion.

5. Let S be a set of sequences of length 15 formed by using the letters a and b such that every pair of sequences in S differ in at least 3 places. What is the maximum number of sequences in S?

Solution. The answer is $2^{11} = 2048$.

We may identify a as 0 and b as 1. Then S is simply the set of binary 15-tuples satisfying the condition that any two tuples in S differ in at least 3 places. For each element s of this set S, there are exactly $15+1=16$ tuples (including itself) that differ from it in at most 1 place. Let B_s denote the set of these tuples. For any distinct $s,t \in S$, we must have $B_s \cap B_t = \emptyset$, otherwise s and t would differ in at most two places. Hence $|S| \cdot 16 \leq 2^{15}$, and so $|S| \leq 2^{11}$. An explicit S meeting the upper bound is then given as follows:

For each of the 2^{11} tuples $(a_1, \ldots, a_{11})$, associate to it the tuple $a = (a_1, \ldots, a_{15})$ in S, where

$$\begin{aligned}
a_{12} &= a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} \pmod 2 \\
a_{13} &= a_2 + a_3 + a_4 + a_8 + a_9 + a_{10} + a_{11} \pmod 2 \\
a_{14} &= a_1 + a_3 + a_4 + a_6 + a_7 + a_{10} + a_{11} \pmod 2 \\
a_{15} &= a_1 + a_2 + a_4 + a_5 + a_7 + a_9 + a_{11} \pmod 2
\end{aligned}$$

To verify that this S is indeed valid, simply check that if any one of the first eleven values of a are changed, then at least two of $a_{12}, a_{13}, a_{14}, a_{15}$ must also be changed, and if any two of the first eleven values of a are changed, then at least one of $a_{12}, a_{13}, a_{14}, a_{15}$ must also be changed.

6. Let n be a positive integer such that the sum of all its positive divisors (inclusive of n) equals to $2n+1$. Prove that n is an odd perfect square.

Solution. Let $n = 2^q r$, where r is odd. Thus $2n+1 = \sigma(n) = \sigma(2^q r) = \sigma(2^q)\sigma(r) = (2^{q+1}-1)\sigma(r)$.

Suppose r is not a perfect square. Let $d_1, d_2, \ldots, d_k$ be all the distinct divisors of r such that $1 \leq d_i < \sqrt{r}, i = 1, 2, \ldots, k$. Note that for every divisor d_i of r, $\frac{r}{d_i}$ is also a divisor of r. Likewise, if d is a divisor of r such that $\sqrt{r} < d \leq r$, then $\frac{r}{d}$ is also a divisor of r. Since $1 \leq \frac{r}{d} < \sqrt{r}$, it follows that $\frac{r}{d}$ is one of $d_1, d_2, \ldots, d_k$. Thus $d_i, \frac{r}{d_i}$, for $i = 1, 2, \ldots, k$ are all the distinct divisors of r. Note that $d_i, \frac{r}{d_i}$ are both odd, so $\sigma(r) = \sum_{i=1}^{k}(d_i + \frac{r}{d_i})$ is even. (Contradiction as $2n+1$ is odd.) Thus r must be a perfect square.

There are two ways to complete.

(i) From $(2^{q+1}-1)\sigma(r) = 2n+1 = 2^{q+1}r+1$, we have $2^{q+1}r \equiv -1 \pmod{2^{q+1}-1}$ and $2^{q+2}r \equiv -2 \pmod{2^{q+1}-1}$. Note that either $2^{q+1}r$ or $2^{q+2}r$ is a perfect square, depending of whether q is odd or even respectively.

$\forall\ q \geq 1, 2^{q+1} - 1 \equiv 3 \pmod 4$, thus $\exists$ some prime $p \equiv 3 \pmod 4$ dividing $2^{q+1}-1$. Note that -1 is a quadratic non-residue $\mod p$, thus $2^{q+1}r$ cannot be a perfect square. Thus $2^{q+2}r$ is a perfect square, and so q is even.

If $q = 0$ then we are done. Suppose $q \geq 2$, then since $2^{q+2}r \equiv -2 \pmod{2^{q+1}-1}$ is a quadratic residue and -1 is a quadratic non-residue $\pmod{2^{q+1}-1}$. In other words, 2 is a quadratic non-residue $\pmod{2^{q+1}-1}$.

It follows that $(-1)^{(\frac{(2^{q+1}-1)^2-1}{8}+1)} = 1 \Leftrightarrow \frac{(2^{q+1}-1)^2-1}{8}$ is odd $\Rightarrow 2^{2q-1} - 2^{q-1}$ is odd. Note that $2q-1 \geq 1$, thus 2^{2q-1} is even so that 2^{q-1} is odd and so can only be 1, which means $q = 1$. (Contradiction since q must be even).

As $q \geq 2$, $2^{q+1} - 1 \equiv 7 \pmod 8$ so that it cannot be a power of 3. Hence there exists a prime factor p of $2^{q+1} - 1$ such that $p \equiv 1, 5, \text{or } 7 \pmod 8$.

(ii) If $q = 0$, then we are done. If not, then from $(2^{q+1} - 1)\sigma(r) = 2n + 1 = 2^{q+1}r + 1$, we have $2^{q+1}r \equiv r \equiv -1 \pmod{2^{q+1} - 1}$. Now $2^{q+1} - 1$ has a prime factor $p \equiv -1 \pmod 4$. Thus $r \equiv -1 \pmod 4$. But r is a perfect square and -1 is not a quadratic residue mod 4, a contradiction. So $q = 0$.

Therefore, n must be an odd perfect square.