Singapore International Mathematical Olympiad Committee

22-9-2001

1. Consider the set of integers $A = \{2^a 3^b 5^c : 0 \leq a, b, c \leq 5\}$. Find the smallest number n such that whenever S is a subset of A with n elements, you can find two numbers p, q in A with $p \mid q$.

2. "Words" are formed with the letters A and B. Using the words x_1, x_2, ..., x_n we can form a new word if we write these words consecutively one next to another: $x_1 x_2 \ldots x_n$. A word is called a palindrome, if it is not changed after rewriting its letters in the reverse order. Prove that any word with 1995 letters A and B can be formed with less than 800 palindromes.

3. Let $n \geq 2$ be an integer and $M = \{1, 2, \ldots, n\}$. For every $k \in \{1, 2, \ldots, n-1\}$, let

$$x_k = \frac{1}{n+1} \sum_{\substack{A \subset M \\ |A|=k}} (\min A + \max A).$$

Prove that $x_1, x_2, \ldots, x_{n-1}$ are integers, not all divisible by 4.

4. The lattice frame construction of $2 \times 2 \times 2$ cube is formed with 54 metal shafts of length 1 (points of shafts' connection are called junctions). An ant starts from some junction A and creeps along the shafts in accordance with the following rule: when the ant reaches the next junction it turns to a perpendicular shaft. t some moment the ant reaches the initial junction A; there is no junction (except for A) where the ant has been twice. What is the maximum length of the ant's path?

5. Let n black and n white objects be placed on the circumference of a circle, and define any set of m consecutive objects from this cyclic sequence to be an m-chain.

(a) Prove that for each natural number $k \leq n$, there exists a chain of $2k$ consecutive pieces on the circle of which exactly k are black.

(b) Prove that there are at least two such chains that are disjoint if

$$k \leq \sqrt{2n+2} - 2$$

.

6. We are given 1999 rectangles with sides of integer not exceeding 1998. Prove that among these 1999 rectangles there are rectangles, say A, B and C such that A will fit inside B and B will fit inside C.

7. We are given N lines ($N > 1$) in a plane, no two of which are parallel and no three of which have a point in common. Prove that it is possible to assign, to each region of the plane determined by these lines, a non-zero integer of absolute value not exceeding N, such that the sum of the integers on either side of any of the given lines is equal to 0.

8. Let S be a set of $2n+1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called *good* if it has 3 points of S on its circumference, $n-1$ points in its interior and $n-1$ in its exterior. Prove that the number of good circles has the same parity as n.

9. Let $ARBPCQ$ be a hexagon. Suppose that $\angle AQR = \angle ARQ = 15^o$, $\angle BPR = \angle CPQ = 30^o$ and $\angle BRP = \angle CQP = 45^o$. Prove that AB is perpendicular to AC.

10. Γ_1 and Γ_2 are two circles on the plane such that Γ_1 and Γ_2 lie outside each other. An external common tangent to the two circles touches Γ_1 at A and Γ_2 at C and an internal common tangent to the two circles touches Γ_1 at B and Γ_2 at D. Prove that the intersection of AB and CD lie on the line joining the centres of Γ_1 and Γ_2

11. Let E and F be the midpoints of AC and AB of $\triangle ABC$ respectively. Let D be a point on BC. Suppose that

(i) P is a point on BF and DP is parallel to CF,

(ii) Q is a point on CE and DQ is parallel to BE,

(iii) PQ intersects BE and CF at R and S respectively.

Prove that $PQ = 3RS$.

12. Let AD, BE and CF be the altitudes of an acute-angled triangle ABC. Let P be a point on DF and K the point of intersection between AP and EF. Suppose Q is a point on EK such that $\angle KAQ = \angle DAC$. Prove that AP bisects $\angle FPQ$.

13. In a square $ABCD$, $\mathcal{C}$ is a circular arc centred at A with radius AB, P and M are points on CD and BC respectively such that PM is tangent to $\mathcal{C}$. Let AP and AM intersect BD at Q and N respectively. Prove that the vertices of the pentagon $PQNMC$ lie on a circle.

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14. In triangle ABC, the angle bisectors of angle B and C meet the median AD at points E and F respectively. If $BE = CF$, prove that $\triangle ABC$ is isosceles.

15. Let $ABCD$ be a convex quadrilateral. Prove that there exists a point E in the plane of $ABCD$ such that $\triangle ABE$ is similar to $\triangle CDE$.

16. Let P, Q be points taken on the side BC of a triangle ABC, in the order B, P, Q, C. Let the circumcircles of PAB, QAC intersect at M $(\neq A)$ and those of PAC, QAB at N $(\neq A)$. Prove that A, M, N are collinear if and only if P, Q are symmetric in the midpoint A' of BC.

17. About a set of four concurrent circles of the same radius r, four of the common tangents are drawn to determine the circumscribing quadrilateral $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral.

18. Three circles of the same radius r meet at common point. Prove that the triangle having the other three points of intersections as vertices has circumradius equal to r

19. For any positive real numbers a, b, c,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

20. Let a, b, c be positive numbers such that $a+b+c \leq 3$. Prove that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \geq \frac{3}{2}.$$

21. Prove that for any positive real numbers a, b, c,

$$\frac{a}{10b+11c} + \frac{b}{10c+11a} + \frac{c}{10a+11b} \geq \frac{1}{7}.$$

22. Prove that for any positive real numbers a, b, c, d, e,

$$\frac{a}{b+2c+3d+4e} + \frac{b}{c+2d+3e+4a} + \frac{c}{d+2e+3a+4b} + \frac{d}{e+2a+3b+4c} + \frac{e}{a+2b+3c+4d} \geq \frac{1}{2}.$$

23. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be n positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Is it true that

$$\frac{a_1^4}{a_1^2+a_2^2} + \frac{a_2^4}{a_2^2+a_3^2} + \cdots + \frac{a_n^4}{a_n^2+a_1^2} \geq \frac{1}{2n}?$$

24. Let a, b, c, d be nonnegative real numbers such that $ab + bc + cd + da = 1$. Prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

25. (**IMO 95**) Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+b)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Cauchy's inequality is useful for these questions.

$$(x_1^2 + x_2^2 + \cdots + x_i^2)(y_1^2 + y_2^2 + \cdots + y_i^2) \geq (x_1y_1 + x_2y_2 + \cdots + x_iy_i)^2.$$

Equality holds iff $x_j = ty_j$ for all j, where t is some constant.

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26. A sequence of natural numbers $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 3$ and

$$a_n = (n+1)a_{n-1} - na_{n-2} \quad (n \geq 2).$$

Find all values of n such that $11|a_n$.

27. Let$\{x_n\}, n \in \mathbb{N}$ be a sequence of numbers satisfying the condition

$$|x_1| < 1, \quad x_{n+1} = \frac{-x_n + \sqrt{3 - 3x_n^2}}{2}, \quad (n \geq 1).$$

(a) What other condition does x_1 need to satisfy so that all the numbers of the sequence are positive?

(b) Is the given sequence periodic?

28. Suppose that a function f defined on the positive integers satisfies $f(1) = 1$, $f(2) = 2$, and

$$f(n+2) = f(n+2 - f(n+1)) + f(n+1 - f(n)), \quad (n \geq 1).$$

(a) Show that $0 \leq f(n+1) - f(n) \leq 1$.

(b) Show that if $f(n)$ is odd, then $f(n+1) = f(n) + 1$.

(c) Determine, with justification, all values of n for which $f(n) = 2^{10} + 1$.

29. Determine the number of all sequences $\{x_1, x_2, \ldots, x_n\}$, with $x_i \in \{a, b, c\}$ for $i = 1, 2, \ldots, n$ that satisfy $x_1 = x_n = a$ and $x_i \neq x_{i+1}$ for $i = 1, 2, \ldots, n-1$.

30. Given is a prime $p > 3$. Set $q = p^3$. Define the sequence $\{a_n\}$ by:

$$a_n = \begin{cases} n & \text{for } n = 0, 1, 2, \ldots, p-1, \\ a_{n-1} + a_{n-p} & \text{for } n > p-1. \end{cases}$$

Determine the remainder when a_q is divided by p.

31. A and B are two candidates taking part in an election. Assume that A receives m votes and B receives n votes, where $m, n \in \mathbb{N}$ and $m > n$. Find the number of ways in which the ballots can be arranged in order that when they are counted, one at a time, the number of votes for A will always be more than that for B at any time during the counting process.

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32. Find all prime numbers p for which the number p^2+11 has exactly 6 different divisors (including 1 and the number itself.)

33. Determine all pairs (a,b) of positive integers such that ab^2+b+7 divides a^2b+a+b.

34. Let p be an odd prime. Prove that

$$1^{p-2}+2^{p-2}+3^{p-2}+\cdots+\left(\frac{p-1}{2}\right)^{p-2} \equiv \frac{2-2^p}{p} \pmod{p}.$$

35. (10th grade) Let $d(n)$ denote the greatest odd divisor of the natural number n. Define the function $f:\mathbb{N}\to\mathbb{N}$ by $f(2n-1)=2^n$, $f(2n)=n+2n/d(n)$ for all $n\in\mathbb{N}$. Find all k such that $f(f(\ldots(1)\ldots))=1997$ where f is iterated k times.

36. Given three real numbers such that the sum of any two of them is not equal to 1, prove that there are two numbers x and y such that $xy/(x+y-1)$ does not belong to the interval $(0,1)$.

37. In the parliament of country A, each MP has at most 3 enemies. Prove that it is always possible to separate the parliament into two houses so that every MP in each house has at most one enemy in his own house.

38. Let $T(x_1, x_2, x_3, x_4) = (x_1 - x_2, x_2 - x_3, x_3 - x_4, x_4 - x_1)$. If x_1, x_2, x_3, x_4 are not equal integers, show that there is no n such that $T^n(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)$.

39. Start with n pairwise different integers x_1, x_2, x_3, x_n, $n > 2$ and repeat the following step:

$$T : (x_1, x_2, \cdots, x_n) \to (\frac{x_1 + x_2}{2}, \frac{x_2 + x_3}{2}, \cdots, \frac{x_n + x_1}{2}).$$

Show that $T, T^2, \cdots$, finally leads to nonintegral component.

40. Is it possible to transform $f(x) = x^2 + 4x + 3$ into $g(x) = x^2 + 10x + 9$ by a sequence of transformations of the form

$$f(x) \to x^2 f(1/x + 1) \quad \text{or} \quad f(x) \to (x-1)^2 f(1/(x-1))?$$

41. Is it possible to arrange the integers $1, 1, 2, 2, \cdots$, 1998, 1998, such that there are exactly $i - 1$ other numbers between any two i's?

42. A rectangular floor can be covered by n 2×2 and m 1×4 tiles, one tile got smashed. Show that one can not substitute that tile by the other type (2×2 or 1×4).

43. In how many ways can you tile a $2 \times n$ rectangle by 2×1 dominoes?

44. In how many ways can you tile a $2 \times n$ rectangle by 1×1 squares and L trominoes?

45. In how many ways can you tile a $2 \times n$ rectangle by 2×2 squares and L trominoes?

46. Let $a_1 = 0$, $|a_2| = |a_1 + 1|, \cdots, |a_n| = |a_{n-1} + 1|$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq -\frac{1}{2}.$$

47. Find the number a_n of all permutations σ of $\{1, 2, \ldots, , n\}$ with $|\sigma(i) - i| \leq 1$ for all i.

48. Can you select from $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ an infinite geometric sequence with sum (a)$\frac{1}{5}$? (b)$\frac{1}{7}$?

49. Let $x_0, a > 0$, $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$. Find $\lim_{n \to \infty} a_n$.

50. Let $0 < a < b$, $a_0 = a$ and $b_0 = b$. For $n \geq 0$, define

$$a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

51. Let $a_0, a_1 = 1, a_n = 2a_{n-1} + a_{n-2}, n > 1$. Show that $2^k | a_n$ if and only if $2^k | n$.

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52. Determine all $\alpha \in \mathbb{R}$ such that there exists a nonconstant function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(\alpha(x+y)) = f(x) + f(y).$$

53. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying

(a) For every $n \in \mathbb{N}, f(n + f(n)) = f(n)$.

(b) For some $n_0 \in \mathbb{N}, f(n_0) = 1$.

Show that $f(n) = 1$ for all $n \in \mathbb{N}$.

54. Suppose that $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(1) = 1$ and for all n,

(a) $3f(n)f(2n+1) = f(2n)(1 + 3f(n))$,

(b) $f(2n) < 6f(n)$.

Find all (k, m) such that $f(k) + f(m) = 2001$.

55. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all $x, y \in \mathbb{R}$,

$$f(x^3 + y^3) = (x+y)((f(x))^2 - f(x)f(y) + (f(y))^2).$$

Prove that for all $x \in \mathbb{R}, f(2001x) = 2001f(x)$.

56. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the property that for all $n \in \mathbb{N}$,

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(f(n))}{f(n+1)}.$$

57. Define $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{Z}$ such that

(a) $f(x, x) = x$,

(b) $f(x, y) = f(y, x)$,

(c) $f(x, y) = f(x, x + y)$,

(d) $g(2001) = 2002$,

(e) $g(xy) = g(x) + g(y) + mg(f(x, y))$.

Determine all integers m for which g exists.

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58. Let $a_0 = 2001$ and $a_{n+1} = \frac{a_n^2}{a_n+1}$. Find the largest integer smaller than or equal to a_{1001}.

59. Given a polynomial $f(x) = x^{100} - 600x^{99} + \cdots$ with 100 real roots and that $f(7) > 1$, show that at least one of the roots is greater than 7.

60. We define $S(n)$ as the number of ones in the binary representation of n. Does there exist a positive integer n such that $\frac{S(n^2)}{S(n)} < \frac{501}{2001}$?

61. A semicircle $\mathcal{S}$ is drawn on one side of a straight line l. C and D are points on $\mathcal{S}$. The tangents to $\mathcal{S}$ at C and D meet l at B and A respectively, with the center of the semicircle between them. Let E be the point of intersection of AC and BD, and F be the point on l such that EF is perpendicular to l. Prove that EF bisects $\angle CFD$.

62. At a round table are 2002 girls, playing a game with n cards. Initially, one girl holds all the cards. In each turn, if at least one girl holds at least two cards, one of these girls must pass a card to each of her two neighbours. The game ends when each girl is holding at most one card.

(a) Prove that if $n \geq 2002$, then the game cannot end.

(b) Prove that if $n < 2002$, then the game must end.

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63. Suppose $\{a_i\}_{i=n}^{m}$ is a finite sequence of integers, and there is a prime p and some k, $n \leq k \leq m$ such that $p|a_k$ but $p \not| a_j$, $n \leq j \leq m, j \neq k$. Prove that $\sum_{i=n}^{m} \frac{1}{a_i}$ cannot be an integer.

64. Prove that for any choice of m and n, $m, n > 1$, $\sum_{i=n}^{m} \frac{1}{i}$ cannot be a positive integer.

65. There is a tournament where 10 teams take part, and each pair of teams plays against each other once and only once. Define a cycle $\{A, B, C\}$ to be such that team A beats team B, B beats C and C beats A. Two cycles $\{A, B, C\}$ and $\{C, A, B\}$ are considered the same. Find the largest possible number of cycles after all the teams have played against each other.

66. Consider the recursions $x_{n+1} = 2x_n + 3y_n$, $y_{n+1} = x_n + 2y_n$ with $x_1 = 2$, $y_1 = 1$. Show that for each integer $n \geq 1$, there is a positive integer K_n such that

$$x_{2n+1} = 2\left(K_n^2 + (K_n + 1)^2\right).$$

67. Suppose that $ABCD$ is a tetrahedron and its four altitudes AA_1, BB_1, CC_1, DD_1 intersect at the point H. Let A_2, B_2, C_2 be points on AA_1, BB_1, CC_1 respectively such that $AA_2 : A_2A_1 = BB_2 : B_2B_1 = CC_2 : C_2C_1 = 2 : 1$. Show that the points H, A_2, B_2, C_2, D_1 are on a sphere.

68. The nonnegative real numbers a, b, c, A, B, C and k satisfy $a+A = b+B = c+C = k$. Prove that $aB + bC + cA \leq k^2$.

69. Find the least constant C such that the inequality

$$x_1x_2 + x_2x_3 + ... + x_{2000}x_{2001} + x_{2001}x_1 \leq C$$

holds for any $\sum_{i=1}^{2001} x_i = 2001$, $x_1, ..., x_{2001} \geq 0$. For this constant C, determine the instances of equality.

70. Let D be a point inside an acute triangle ABC such that

$$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA = AB \cdot BC \cdot CA.$$

Determine the geometric position of D.

71. Consider the polynomial $p(x) = x^{2001} + a_1x^{2000} + a_2x^{1999} + ... + a_{2000}x + 1$. where all the a_i's are nonnegative. If the equation $p(x) = 0$ has 2001 real roots, prove that $p(2001) \geq 2002^{2001}$.

72. Let a, b, c, d be nonnegative real numbers such that $a + b + c + d = 1$. Prove that $bcd + cda + dab + abc \leq \frac{1}{27} + \frac{176}{27}abcd$. Determine when equality holds.

73. A sequence $\{a_k\}$ satisfies the following conditions:

(a) $a_0 = \frac{1}{2001}$,

(b) $a_{k+1} = a_k + \frac{1}{n}a_k^2$, $k = 0, 1, 2, ..., n$.

Prove that $1 - \frac{1}{2000n} < 2000a_n < 1$.

74. Let ABC be an equilateral triangle. Draw the semicircle Γ which has BC as its diameter, where Γ lies on the opposite side of BC as A. Show that the straight lines drawn from A that trisect the line BC also trisect Γ when they are extended.

75. ABC is an equilateral triangle. P is the midpoint of arc AC of its circumcircle, and M is an arbitrary point of the arc. N is the midpoint of BM and K is the foot of the perpendicular from P to MC. Prove that ANK is an equilateral triangle.

76. Two concentric circles are given with a common centre O. From a point A on the outer circle, two tangents to the inner circle are drawn, meeting the latter at D and E respectively. AD and ED are extended to meet the outer circle at C and B respectively. Show that $(\frac{AB}{BC})^2 = \frac{BE}{BD}$.

77. Ali Baba the carpet merchant has a rectangular piece of carpet whose dimensions are unknown. Unfortunately, his tape measure is broken and he has no other measuring instruments. However, he finds that if he lays it flat on the floor of either of his storerooms, then each corner of the carpet touches a different wall of that room. He knows that the sides of the carpet are integral numbers of feet, and that his two storerooms have the same (unknown) length, but widths of 38 and 50 feet respectively. What are the carpet's dimensions?

78. Let I be the incentre of the non-isosceles triangle ABC. Let the incircle of ABC touch the sides BC, CA and AB at the points A_1, B_1 and C_1 respectively. Prove that the circumcentres of ΔAIA_1, ΔBIB_1 and ΔCIC_1 are collinear.

Solutions for 22-9-2001

1. Consider the set of integers $A = \{2^a3^b5^c : 0 \le a, b, c \le 5\}$. Find the smallest number n such that whenever S is a subset of A with n elements, you can find two numbers p, q in A with $p \mid q$.

Soln. Identify a number $2^a3^b5^c$ in A by its triple of indices $\overline{abc}$. Thus $\overline{000}$ represents $2^03^05^0$ while $\overline{213}$ represents $2^23^15^3$. Consider the following arrangement of numbers into 16 rows.

$\overline{000}$							
$\overline{100}$							
$\overline{200}$	$\overline{110}$						
$\overline{300}$	$\overline{111}$	$\overline{210}$	$\overline{120}$				
$\overline{400}$	$\overline{211}$	$\overline{310}$	$\overline{130}$	$\overline{220}$			
$\overline{500}$	$\overline{311}$	$\overline{410}$	$\overline{140}$	$\overline{320}$	$\overline{302}$		$\overline{221}$
$\overline{510}$	$\overline{411}$	$\overline{420}$	$\overline{240}$	$\overline{321}$	$\overline{312}$	$\overline{330}$	$\overline{222}$
$\overline{520}$	$\overline{511}$	$\overline{430}$	$\overline{340}$	$\overline{421}$	$\overline{412}$	$\overline{331}$	$\overline{322}$
$\overline{530}$	$\overline{521}$	$\overline{440}$	$\overline{341}$	$\overline{431}$	$\overline{512}$	$\overline{332}$	$\overline{422}$
$\overline{540}$	$\overline{531}$	$\overline{441}$	$\overline{342}$	$\overline{432}$	$\overline{513}$	$\overline{333}$	$\overline{522}$
$\overline{550}$	$\overline{541}$	$\overline{442}$	$\overline{352}$	$\overline{532}$	$\overline{514}$	$\overline{433}$	
$\overline{551}$	$\overline{542}$	$\overline{443}$	$\overline{353}$		$\overline{524}$		
$\overline{552}$	$\overline{543}$	$\overline{444}$			$\overline{534}$		
$\overline{553}$	$\overline{544}$						
$\overline{554}$							
$\overline{555}$							

Turn the first column in 6 by considering the six permutations and turn each of the other columns into 3 by considering the cyclic permutations. Note that (i) every number is one of the rows, (ii) every two numbers in the same row do not divide each other (iii) for every two numbers in the same column, one must divided the other. Since the row with the most elements is 27, the answer is 28.

(Joel's soln) The problem is equivalent to the following: Consider the ordered triples of the form $\{(a, b, c) : 0 \le a, b, c \le 5\}$. Find the smallest number n such that if there are n such triples, you can always find two (p, q, r) and (x, y, z) such that $p \ge x$, $q \ge y$, and $r \ge z$. If this is the case, we say that the two triples are *comparable*.

First we note that any two triples are comparable if their second and third elements are the same. Let A be a set of pairwise noncomparable triples. Since there are 36 possible combinations for (b, c), A has at most 36 triples. Examine the sequence of 11 triples in order:

$$(a, 0, 0), (a, 0, 1), (a, 0, 2), (a, 0, 3), (a, 0, 4), (a, 0, 5), (a, 1, 5), (a, 2, 5), (a, 3, 5), (a, 4, 5), (a, 5, 5).$$

For these to be in A, the first elements have to be strictly decreasing. Thus at most 6 of these can be in A or at least 5 cannot be in A. Similarly, at least three of the following triples

$$(a,1,0),(a,2,0),(a,3,0),(a,4,0),(a,5,0),(a,5,1),(a,5,2),(a,5,3),(a,5,4)$$

and at least 1 of the following triples:

$$(a,1,1),(a,1,2),(a,1,3),(a,1,4),(a,2,4),(a,3,4),(a,4,4)$$

are not in A. Thus A can have at most 27 triples and if 28 triples are chosen, then at least two of them are comparable. Below is a collection of 27 pairwise noncomparable triples (the sum of the 3 components is 7):

$$\begin{aligned}
&(0,2,5),(0,3,4),(0,4,3),(0,5,2),(1,1,5),(1,2,4),(1,3,3),\\
&(1,4,2),(1,5,1),(2,0,5),(2,1,4),(2,2,3),(2,3,2),(2,4,1),\\
&(2,5,0),(3,0,4),(3,1,3),(3,2,2),(3,3,1),(3,4,0),(4,0,3),\\
&(4,1,2),(4,2,1),(4,3,0),(5,0,2),(5,1,1),(5,2,0)
\end{aligned}$$

So the answer is 28.

2. (Byelorrussian MO 95) "Words" are formed with the letters A and B. Using the words x_1, x_2, ..., x_n we can form a new word if we write these words consecutively one next to another: $x_1x_2\dots x_n$. A word is called a palindrome, if it is not changed after rewriting its letters in the reverse order. Prove that any word with 1995 letters A and B can be formed with less than 800 palindromes.

Soln. *Official solution*: (The key idea is to find the longest word that can be formed using at most 2 palindromes.) First of all, it is easy to check that any 5-letter word may be formed with at most two palindromes. Indeed, (A and B are symmetric).

$AAAAA = AAAAA$, $AAAAB = AAAA + B$, $AAABA = AA + ABA$,
$AAABB = AAA + BB$, $AABAA = AABAA$, $AABAB = AA + BAB$,
$AABBA = A + ABBA$, $AABBB = AA + BBB$, $ABAAA = ABA + AA$,
$ABAAB = A + BAAB$, $ABABA = ABABA$, $ABABB = ABA + BB$,
$ABBAA = ABBA + A$, $ABBAB = ABBA + B$, $ABBBA = ABBBA$,
$ABBBB = A + BBBB$.

Let us consider an arbitrary word with 1995 letters and divide it into words with 5 letters each. Each of these $1995/5 = 399$ words may be formed with at most two palindromes. Thus any 1995-letter word may be formed with at most $399 \times 2 = 798$ palindromes.

3. (49th Romania National MO 1998) (10th Form) Let $n \geq 2$ be an integer and $M = \{1, 2, \dots, n\}$. For every $k \in \{1, 2, \dots, n-1\}$, let

$$x_k = \frac{1}{n+1} \sum_{\substack{A \subset M \\ |A|=k}} (\min A + \max A).$$

Prove that $x_1, x_2, \dots, x_{n-1}$ are integers, not all divisible by 4.

Soln. For each $i \in \{1, 2, \dots, n\}$, there are $\binom{n-i}{k-1}$ subsets, each with k elements and contains i as the minimum element. (Note that $\binom{n}{k} = 0$ if $k > n$.) Also there are $\binom{i-1}{k-1}$ subsets, each with k elements and contains i as the maximum element. Thus

$$\begin{aligned} x_k &= \frac{1}{n+1} \sum_{\substack{A \subset M \\ |A|=k}} (\min A + \max A) \\ &= \frac{1}{n+1} \left[1\binom{n-1}{k-1} + 2\binom{n-2}{k-1} + \cdots + (n+1-k)\binom{k-1}{k-1} \right. \\ &\quad \left. + n\binom{n-1}{k-1} + (n-1)\binom{n-2}{k-1} + \cdots + k\binom{k-1}{k-1} \right] \\ &= \binom{n}{k} \end{aligned}$$

Thus $x_1, \dots, x_{n-1}$ are all integers. Since $x_1 + \cdots + x_{n-1} = \binom{n}{1} + \cdots + \binom{n}{n-1} = 2^n - 2$, not all of $x_1, \dots, x_{n-1}$ are divisible by 4.

(2nd Solution) For each k element subset $A = \{a_1, \dots, a_k\}$ with $a_i < a_j$ if $i < j$, with $a_1 = 1 + p$ and $a_k = n - q$, let $B = \{b_1, \dots, b_k\}$, where $b_i = a_i + q - p$. We have $\min A + \min B + \max A + \max B = 2n + 2$. Since the number of k element subsets is $\binom{n}{k}$, we have

$$x_k = \frac{1}{n+1} \sum_{\substack{A \subset M \\ |A|=k}} (\min A + \max A) = \frac{1}{n+1} \binom{n}{k} \frac{2n+2}{2} = \binom{n}{k}.$$

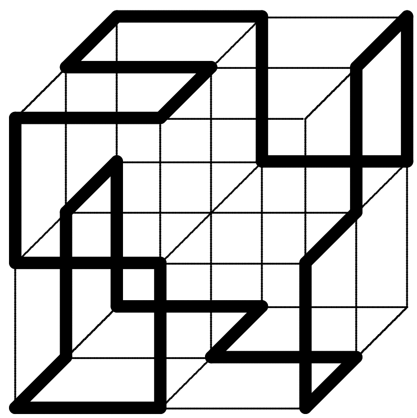
Here's a sketch/hint for 3: For each set A, consider the set A':

$$A' = \{n + 1 - x | x \in A\}$$

Then, $\min A = n + 1 - \max A'$. Do the usual summation rewriting, and you can compute x_k explicitly to be some binomial function. The other bit follows readily too using another standard trick.

4. (Byelorrussian MO 95) The lattice frame construction of $2 \times 2 \times 2$ cube is formed with 54 metal shafts of length 1 (points of shafts' connection are called junctions). An ant starts from some junction A and creeps along the shafts in accordance with the following rule: when the ant reaches the next junction it turns to a perpendicular shaft. At some moment the ant reaches the initial junction A; there is no junction (except for A) where the ant has been twice. What is the maximum length of the ant's path?

Soln. *Official solution*: The maximum length of an ant's path is equal to 24. First we prove that the path along which the ant creeps, has at most 24 junctions of the shafts of the cube frame. By the condition, any two consecutive shafts in the path (except possibly for the first and the last shafts) are perpendicular. In particular, the ant's path passes at most one shaft on all but one edge of the cube. Thus there are at most 13 shafts along the edges of the cube. However, each vertex in the path requires two shafts. Thus the path misses at least two vertices of the cube. Hence the ant's path passes through at most 25 junctions of the shafts. The ant's path consists of an even number of junctions. This is easily seen to be true by taking the starting point as the origin and the three mutually perpendicular lines passing through it as the axes and assume that each shaft is of unit length. Then each move by the ant causes exactly a change of one unit in one of the coordinates. Thus the total number of moves is even. Thus the length of the path is at most 24. A path of length 24 is shown in the figure.



5. Let n black and n white objects be placed on the circumference of a circle, and define any set of m consecutive objects from this cyclic sequence to be an m-chain.

(a) Prove that for each natural number $k \leq n$, there exists a chain of $2k$ consecutive pieces on the circle of which exactly k are black.

(b) Prove that there are at least two such chains that are disjoint if

$$k \leq \sqrt{2n+2} - 2$$

.

Soln. (a) Label the positions $1, 2, \ldots, 2n$ in the clockwise direction. For each i, if the object is i is black, let $a_i = 1$. Otherwise, let $a_i = -1$. Fix $k \leq n$. Define the function $g(i) = \sum_{j=i}^{i+2k-1} a_j$. Then there is a chain of $2k$ consecutive pieces of which exactly k are black iff $g(i) = 0$ for some i. Since $\sum_{i=1}^{2n} g(i) = 0$, and since $g(i+1) - g(i) = \pm 2, 0$, such a $g(i)$ exists.

(b): Suppose on the contrary that there does not exist 2 disjoint chains. WLOG, we may assume from (a) that $g(2k) = 0$ and $g(2k+1), \ldots g(2n)$ are all nonzero, and by the same continuity argument, they must all be of the same sign. WLOG, assume $g(2k+1), \ldots, g(2n) < 0$ and thus $g(2k), \ldots, g(2n-1) \leq -2$. Therefore, we have:

$$g(1) + \ldots + g(2k) = -(g(2k+1) + \ldots + g(2n)) \geq 2(2n-2k)$$

Thus, the sum on the left is maximum when $g(1) = 0, g(1) = 2, \ldots, g(k) = g(k+1) = 2(k-1), \ldots, g(2k) = 0$, that is, $g(1), \ldots, g(k)$ forms an increasing arithmetic progression with a difference of 2, and $g(k+1), \ldots, g(2k)$ forms a decreasing arithmetic progression with a difference of -2. This yields:

$$2(k-1)^2 \geq 2(2n-2k) \Longrightarrow k^2 + 4k + 4 > 2n + 2$$

which yields the required result.

Solutions for 29-9-2001

6. We are given 1999 rectangles with sides of integer not exceeding 1998. Prove that among these 1999 rectangles there are rectangles, say A, B and C such that A will fit inside B and B will fit inside C.

Soln. *(T& T Spring 1989 Q6, modified slightly).* We partition the given set of rectangles into 999 suitably chosen (pairwise disjoint) subsets S_1, …, S_{999}. These are defined as follows:

For each $i \in \{1, \ldots, 999\}$, let S_i be the set of rectangles which have the following properties:

(a) the shorter side has length at least i,

(b) the longer side has length at most $1998 - i$,

(c) either the shorter side has length i or the longer side has length $1998 - i$.

Any three rectangles in S_i can always be arranged with the desired inclusion property. Thus the result follows by the pigeon hole principle.

7. We are given N lines ($N > 1$) in a plane, no two of which are parallel and no three of which have a point in common. Prove that it is possible to assign, to each region of the plane determined by these lines, a non-zero integer of absolute value not exceeding N, such that the sum of the integers on either side of any of the given lines is equal to 0.

Soln. *(Tournament of Towns Spring 1989 Q5).* First note the regions can be painted in two colours so that two regions sharing a common side have different colours. This is trivial for $N = 1$. Assume that it is true for $N = k$. When the $(k+1)$st line is drawn we simply reverse the colours on exactly one side of this line.

Assign to each region an integer whose magnitude is equal to the number of vertices of that region. The sign is + is the region is one of the two colours and − if it is the other colour. Let L be one of the given lines. Consider an arbitrary vertex on one side of L. If this vertex is on L, then it contributes $= 1$ to one region and -1 to a neighbouring region. If it is not on L, it contributes $= 1$ to two regions and -1 to two regions. Thus the sum of the numbers of one side of L is 0.

8. (APMO 99) Let S be a set of $2n+1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called *good* if it has 3 points of S on its circumference, $n-1$ points in its interior and $n-1$ in its exterior. Prove that the number of good circles has the same parity as n.

Soln. For any two points A and B, let P_1, P_2, …, P_k be points on one side of the line AB and P_{k+1}, …, P_{2n-1} be points on the other side. We shall prove that the number of good circles passing through A and B is odd. Let

$$\theta_i = \begin{cases} \angle AP_iB & \text{if } i = 1, \ldots, k \\ 180^\circ - \angle AP_iB & \text{if } i = k+1, \ldots, 2n-1 \end{cases}$$

It is easy to see that P_j is in the interior of the circle ABP_i, if and only if

$$\begin{cases} \theta_j > \theta_i & \text{for } 1 \le j \le k \\ \theta_j < \theta_i & \text{for } k+1 \le j \le 2n-1 \end{cases}.$$

Arrange the points P_i in increasing order of their corresponding angles θ_i. Colour the points P_i, $i = 1, \ldots, k$, black and the points P_i, $i = k+1, \ldots, 2n-1$, white. For any point X (different from A and B), let B_X be the number of black points less than X minus the number of black points greater than X and W_X be the corresponding difference for white points. (Note that black points which are greater than X are interior points of the circle ABX while the white points greater than X are exterior points.) Define $D_X = B_X - W_X$. From the forgoing discussion we know that $\triangle ABX$ is good if and only if $D_X = 0$. We call such a point *good*. If $X < Y$ are consecutive points, then $D_X = D_Y$ if X and Y are of different colours. (It is easy to show that $D_Y - D_X = -2$ if X and Y are both white and $D_Y - D_X = 2$ if X and Y are both black. But we do not need these.)

If all the points are of the same colour, there is only one good point, namely the middle point among the P_i's.

Now we suppose that there are points of either colour. Then there is a pair of adjacent points, say X, Y, with different colour. Since $D_X = D_Y$, either both are good or both are not good. Their removal also does not change the value of D_Z for any other point Z. Thus the removal of a pair of adjacent points of different colour does not change the parity of the number of good points. Continue to remove such pairs until only points of the same colour are left. When this happens there is only one good point. Thus the number of good circles through A and B is odd.

Now let g_{AB} be the number of good circles through A and B. Since each good circle contains exactly three points, i.e., three pairs of points. Then $\sum g_{AB} = 3g$ where g is total number of good circles. Since there are a total of $n(2n+1)$ terms in the sum, and each term is odd, we have $g \equiv n \pmod 2$.

Solutions for 27-10-2001

19. For any positive real numbers a, b, c,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Soln. Let $x = a+b$, $y = a+c$, $z = b+c$, then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x} + \frac{x}{z} + \frac{z}{x} + \frac{y}{z} + \frac{z}{y} - 3\right) \geq \frac{3}{2}.$$

Soln 2.

$$\left[\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right][a(b+c) + b(c+a) + c(a+b)] \geq (a+b+c)^2.$$

Thus

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{(a+b+c)^2}{2(ab+ac+bc)}.$$

Also

$$(a+b+c)^2 = (a^2+b^2+c^2) + 2(ab+bc+ac) \geq 3(ab+bc+ac).$$

Thus

$$\frac{(a+b+c)^2}{a(b+c)+b(c+a)+c(a+b)} \geq \frac{3}{2}.$$

Soln 3. Since the inequality is symmetric about a, b, c, we may assume that $a \geq b \geq c > 0$. Then,

$$\frac{1}{b+c} \geq \frac{1}{a+c} \geq \frac{1}{a+b}.$$

By rearrangement inequality, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{a}{c+a}+\frac{b}{a+b}+\frac{c}{b+c},$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}.$$

The inequality follows by adding these two inequalities.

20. Let a, b, c be positive numbers such that $a+b+c \leq 3$. Prove that

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} \geq \frac{3}{2}.$$

Soln. Apply $AM \geq HM$ on the three numbers $a+1, b+1, c+1$ we have

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} \geq \frac{9}{(1+a)+(1+b)+(1+c)} \geq \frac{3}{2}.$$

Soln 2. Using Cauchy's inequality and denoting the denominators by x, y, z:

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)(x+y+z) \geq 9.$$

Thus

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{9}{3+a+b+c} \geq \frac{3}{2}.$$

21. Prove that for any positive real numbers a, b, c,

$$\frac{a}{10b+11c}+\frac{b}{10c+11a}+\frac{c}{10a+11b} \geq \frac{1}{7}.$$

Soln. We shall prove a more general inequality:

$$\frac{a}{mb+nc}+\frac{b}{mc+na}+\frac{c}{ma+nb} \geq \frac{3}{m+n},$$

where m, n are positive real numbers.

Write the denominators as A, B, C respectively. By Chauchy-Schwarz inequality, we have

$$\left[\frac{a}{A}+\frac{b}{B}+\frac{c}{C}\right][aA+bB+cC] \geq (a+b+c)^2.$$

Since $aA + bB + cC = (m+n)(ab+bc+ca)$ and $3(ab+bc+ca) \le (a+b+c)^2$, the result follows.

Soln 2. The inequality can also be proved by clearing the denominator and moving all terms to the left hand side. After some simplification, the resulting inequality to be proved is equivalent to

$$770(a^3+b^3+c^3) - 253(a^2b+b^2c+c^2a) - 510(a^2c+b^2a+c^2b) - 21abc \ge 0.$$

By rearrangement inequality, we have $a^3+b^3+c^3 \ge a^2b+b^2c+c^2a$ and $a^3+b^3+c^3 \ge a^2c+b^2a+c^2b$. Using these and $a^3+b^3+c^3 \ge 3abc$, the above inequality follows.

22. Prove that for any positive real numbers a_i, $i = 1, \ldots 5$,

$$\sum_{i=1}^{5} \frac{a_i}{a_{i+1} + 2a_{i+2} + 3a_{i+3} + 4a_{i+4}} \ge \frac{1}{2}$$

where the subscripts are to be taken mod 5.

Soln. Use the same method as in the previous problem. The only extra thing that you have to note is that

$$\sum a_i^2 \ge \frac{1}{2} \sum_{1 \le i < j \le 5} a_i a_j.$$

23. (SMO2001) Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be n positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Is it true that

$$\frac{a_1^4}{a_1^2+a_2^2} + \frac{a_2^4}{a_2^2+a_3^2} + \cdots + \frac{a_n^4}{a_n^2+a_1^2} \ge \frac{1}{2n}?$$

Soln. The answer is yes. First observe that

$$\left(\frac{a_1^4}{a_1^2+a_2^2} + \frac{a_2^4}{a_2^2+a_3^2} + \cdots + \frac{a_n^4}{a_n^2+a_1^2}\right) - \left(\frac{a_2^4}{a_1^2+a_2^2} + \frac{a_3^4}{a_2^2+a_3^2} \cdots + \frac{a_1^4}{a_n^2+a_2^2}\right) = 0.$$

Thus

$$\begin{aligned}
&\frac{a_1^4}{a_1^2+a_2^2} + \frac{a_2^4}{a_2^2+a_3^2} + \cdots + \frac{a_n^4}{a_n^2+a_1^2} \\
&= \frac{1}{2}\left(\frac{a_1^4+a_2^4}{a_1^2+a_2^2} + \frac{a_2^4+a_3^4}{a_2^2+a_3^2} + \cdots + \frac{a_n^4+a_1^4}{a_n^2+a_1^2}\right) \\
&\ge \frac{1}{4}[(a_1^2+a_2^2) + \cdots + (a_n^2+a_1^2)] \\
&= \frac{1}{2}(a_1^2 + \cdots + a_n^2) \ge \frac{1}{n}
\end{aligned}$$

which completes the proof. The last inequality follows as

$$\sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + \cdots + a_n}{n} = \frac{1}{2n}.$$

Soln 2. Denote the denominators by $A_1, A_2, \ldots, A_n$. Then

$$\left[\frac{a_1^4}{A_1} + \frac{a_2^4}{A_2} + \cdots + \frac{a_n^4}{A_n}\right][A_1 + A_2 + \cdots + A_n] \geq (a_1^2 + a_2^2 + \cdots + a_n^2)^2.$$

But

$$A_1 + A_2 + \cdots + A_n = 2(a_1^2 + \cdots + a_n^2).$$

Thus we only need to prove

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{1}{n}.$$

24. Let a, b, c, d be nonnegative real numbers such that $ab + bc + cd + da = 1$. Prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

Soln. (NT-ST 2001, IMO proposed 1991) By Cauchy-Schwarz Inequality,

$$a^2 + b^2 + c^2 + d^2 \geq ab + bc + cd + da = 1.$$

Assume without loss of generality that $a \geq b \geq c \geq d \geq 0$. Then

$$\frac{1}{b+c+d} \geq \frac{1}{a+c+d} \geq \frac{1}{a+b+d} \geq \frac{1}{a+b+c} > 0.$$

Using Chebyshev's Inequality twice followed by $AM \geq GM$, and writing $x = b+c+d, y = c+d+a, z = d+a+b, w = a+b+c$, we have

$$\begin{aligned}
\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} + \frac{d^3}{w} &\geq \frac{a^3+b^3+c^3+d^3}{4}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right) \\
&\geq \frac{(a^2+b^2+c^2+d^2)(a+b+c+d)}{16}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right) \\
&\geq \frac{a+b+c+d}{16}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right) \\
&\geq \frac{(x+y+z+w)}{3 \times 16}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right) \\
&\geq \frac{4(xyzw)^{\frac{1}{4}}}{3 \times 16}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right) \\
&\geq \frac{4 \times 4(xyzw)^{\frac{1}{4}}}{3 \times 16}\frac{1}{(xyzw)^{\frac{1}{4}}} \geq \frac{1}{3}
\end{aligned}$$

Soln 2. Let the denominators be A, B, C, D. Then

$$\left[\frac{a^3}{A} + \cdots + \frac{d^3}{D}\right][aA + \cdots + dD] \geq (a^2 + \cdots + d^2)^2.$$

Thus we need to prove

$$\frac{(a^2 + \cdots + d^2)^2}{2 + 2ac + 2bd} \geq \frac{1}{3}.$$

Since $2ac \leq a^2 + c^2$ and $2bd \leq b^2 + d^2$, we need to prove

$$\frac{x^2}{2+x} \geq \frac{1}{3}, \quad \text{or} \quad 3x^2 - x - 2 = (3x+2)(x-1) \geq 0$$

which is obviously true since $x = a^2 + \cdots + d^2 \geq 1$.

25. (**IMO 95**) Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+b)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Soln. Let S be the right hand side and $T = a(b+c)+b(a+c)+c(a+b) = 2(ab+ac+bc) \geq 6(abc)^{2/3} = 6$. Then by Cauchy's inequality, we have

$$ST \geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 = \frac{T^2}{4}.$$

Thus $S \geq T/4 \geq 3/2$ as desired.

Second Solution Let $x = 1/a$. $y = 1/b$, $z = 1/c$. Then $xyz = a$ and

$$S = \frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}.$$

By Cauchy's inequality,

$$[(x+y) + (z+x) + (x+y)]S \geq (x+y+z)^2$$

or $S \geq \frac{(x+y+z)}{2} \geq \frac{3}{2}(xyz)^{1/3} = \frac{3}{2}$ as desired.

Solutions for 3-11-2001

26. A sequence of natural numbers $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 3$ and

$$a_n = (n+1)a_{n-1} - na_{n-2} \quad (n \geq 2).$$

Find all values of n such that $11|a_n$.

Soln. Rearranging the recurrence relation, we obtain: $a_n - a_{n-1} = n(a_{n-1} - a_{n-2})$. Substituting $b_n = a_n - a_{n-1}$, we find that $b_n = nb_{n-1} = n(n-1)b_{n-2} = \ldots = n!$. Hence, $a_n = 1! + 2! + \ldots + n!$. One can easily verify that $11|a_4, a_8, a_{10}$. Since $11|a_{10}$ and $11|i!$ for all $i \geq 11$, this implies that $11|a_n$ for all $n \geq 10$. So the required answers are: $n = 4, 8$ and $n \geq 10$.

27. Let$\{x_n\}, n \in \mathbb{N}$ be a sequence of numbers satisfying the condition

$$|x_1| < 1, \quad x_{n+1} = \frac{-x_n + \sqrt{3 - 3x_n^2}}{2}, \quad (n \geq 1).$$

(a) What other condition does x_1 need to satisfy so that all the numbers of the sequence are positive?

(b) Is the given sequence periodic?

Soln.

(a) For the sequence to be positive, we require $3 - 3x_n^2 \geq x_n^2$, ie that $|x_n| \leq \frac{\sqrt{3}}{2}$. One can easily check by plugging into the recurrence relation that when $|x_n| \leq \frac{\sqrt{3}}{2}$, then $|x_{n+1}| \leq \frac{\sqrt{3}}{2}$ also, so the entire sequence will be positive. Hence, the required condition is $|x_1| \leq \frac{\sqrt{3}}{2}$.

(b) Substitute $x_n = \sin\theta_n$ into the recurrence relation. After some simplification, we find that $x_{n+1} = -\frac{1}{2}\sin\theta_n + \frac{\sqrt{3}}{2}\cos\theta_n = sin(\frac{\pi}{3} - \theta_n)$. Hence, $x_{n+2} = x_n$ and the sequence is periodic.

28. Suppose that a function f defined on the positive integers satisfies $f(1) = 1$, $f(2) = 2$, and

$$f(n+2) = f(n+2-f(n+1)) + f(n+1-f(n)), \quad (n \geq 1).$$

(a) Show that $0 \leq f(n+1) - f(n) \leq 1$.

(b) Show that if $f(n)$ is odd, then $f(n+1) = f(n) + 1$.

(c) Determine, with justification, all values of n for which $f(n) = 2^{10} + 1$.

Soln. The key is to show that given $n = 2^k + m$, where $0 \leq m \leq 2^k - 2$, then $f(n) = f(m+1) + 2^{k-1}$. Also, if $n = 2^k + 2^k - 1 = 2^{k+1} - 1$, then $f(n) = 2^k$. This can be done using strong induction on k. Notice that this helps us to express all the values of $f(n)$ from $n = 2^k$ to 2^{k+1}, in terms of the values of $f(n)$ from $n = 0$ to $n = 2^k - 1$. (We can define

$f(0) = 0$ for convenience; this definition is consistent with the recurrence relation.) Once we have established this claim, parts (a) and (b) follow essentially by induction. Also, the answer to (c) is $n = 2^{11}$. (Endnote: The key to solving this question would be to list out the values and try to find a pattern, which can then be rigorously justified using induction.)

29. Determine the number of all sequences $\{x_1, x_2, \ldots, x_n\}$, with $x_i \in \{a, b, c\}$ for $i = 1, 2, \ldots, n$ that satisfy $x_1 = x_n = a$ and $x_i \neq x_{i+1}$ for $i = 1, 2, \ldots, n-1$.

Soln. Let the number of sequences for which $x_1 = x_n = a$ be A_n. Define B_n to be the number of sequences with $x_1 = a, x_n = b$, and let C_n be the number of sequences with $x_1 = a, x_n = c$. Notice by symmetry that $B_n = C_n$ (just simply switch the b's and c's in the sequence). Notice that $A_n = B_{n-1} + C_{n-1} = 2B_{n-1}$, since any sequence of length n that ends with a is generated by a sequence of length $n-1$ which ends with b or c. (Conversely, given a sequence of length $n-1$ that ends with b or c, we can simply append a to obtain a sequence of length n which ends with a.) Similarly, $B_n = A_{n-1} + C_{n-1} = A_{n-1} + B_{n-1}$, since any sequence of length n that ends with b must be generated by a sequence of length $n-1$ generated with a or c. Having established this simultaneous system of recurrence relations, we substitute and solve for $B_n = B_{n-1} + 2B_{n-2}$, $B_1 = 0, B_2 = 1$. Solving, we obtain that $A_n = 2B_{n-1} = \frac{2}{3}[\ 2^{n-2} + (-1)^{n-1}]$.

30. Given is a prime $p > 3$. Set $q = p^3$. Define the sequence $\{a_n\}$ by:

$$a_n = \begin{cases} n & \text{for } n = 0, 1, 2, \ldots, p-1, \\ a_{n-1} + a_{n-p} & \text{for } n > p-1. \end{cases}$$

Determine the remainder when a_q is divided by p.

Soln. (This remains an open question. I suspect the key is to prove that $a_{p^2-1} \equiv a_0 \pmod{p}$, from which it follows that $a_{p^3} \equiv p-1 \pmod{p}$.)

31. A and B are two candidates taking part in an election. Assume that A receives m votes and B receives n votes, where $m, n \in \mathbb{N}$ and $m > n$. Find the number of ways in which the ballots can be arranged in order that when they are counted, one at a time, the number of votes for A will always be more than that for B at any time during the counting process.

Soln. We model the vote counting process on the Cartesian plane. Starting at $(0, 0)$, we move to $(1, 1)$ if the first vote goes to A, else we move to $(1, -1)$ if the first vote goes to B instead. Notice that this walk will end at $(m+n, m-n)$. The number of ways of counting in which the number of votes A gets is always more than that which B has is equal to the number of paths on the plane from $(0, 0)$ to $(m+n, m-n)$ that do not cross the x-axis. We count the complement of this set, ie the number of paths that do cross the x-axis. Notice that the first vote must go to A. Starting from $(1, 1)$, the number of paths that cross the

x-axis and end at $(m+n, m-n)$ is equal to the number of paths (unrestricted) from $(1,-1)$ to $(m+n, m-n)$. To see this, take any given path from $(1,1)$ to $(m+n, m-n)$ and let X be the first point of intersection with the x-axis. Reflect the portion of this path to the left of X about the x-axis and we have a path from $(1,-1)$ to $(m+n, m-n)$. Vice versa, we can take any such path from $(1,-1)$ to $(m+n, m-n)$ and reflect it back uniquely. The number of paths from $(1,-1)$ to $(m+n, m-n)$ is $\binom{m+n-1}{m}$, since you need to choose m steps upwards out of a possible $m+n-1$. Similarly, the number of paths from $(1,1)$ to $(m+n, m-n)$ is $\binom{m+n-1}{m-1}$. Hence, the number of paths from $(0,0)$ to $(m+n, m-n)$ that do not cross the x-axis is $\binom{m+n-1}{m-1} - \binom{m+n-1}{m} = \frac{m-n}{m+n}\binom{m+n}{m}$.

Solutions for 10-11-2001

32. (Auckland MO 98) Find all prime numbers p for which the number p^2+11 has exactly 6 different divisors (including 1 and the number itself.)

Soln. If $p = 3$, $p^2 + 11$ has exactly 6 divisors. Now let $p > 3$. Then p is odd and $p^2 \equiv 1 \pmod 4$, thus $p^2 + 11 \equiv 0 \pmod 4$. Also $p^2 \equiv 1 \pmod 3$, thus $p^2 + 11 \equiv 0 \pmod 3$. Thus $p^2 + 11 \equiv 0 \pmod{12}$. Since every $p^2 + 11 > 12$ and every divisor of 12 is also a divisor of p^2+11, it follows that p^2+11 has more divisors than 12. Thus p^2+11 has more than 6 divisors. The only prime number with the desired property is therefore 3.

33. (IMO 98) Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

Soln. Since $ab^2 + b + 7 | b(a^2b + a + b)$ and $a^2b^2 + ab + b^2 = a(ab^2 + b + 7) + (b^2 - 7a)$, we have either $b^2 - 7a = 0$ or $b^2 - 7a$ is a multiple of $ab^2 + b + 7$. The former implies that $b = 7t$ and $a = 7t^2$. Indeed these are solutions for all positive t.

For the second case, we note that $b^2 - 7a < ab^2 + b + 7$. Thus $b^2 - 7a < 0$. If $b \geq 3$, then $ab^2 + b + 7 > 7a - b^2$. Thus for $ab^2 + b + 7$ to divide $7a - b^2$, $b = 1, 2$. The case $b = 1$ requires that $7a - 1$ be divisible by $a + 8$. The quotients are less than 7. Testing each of the possibilities yields $a = 49, 11$. These are indeed solutions.

The case $b = 2$ requires that $7a - 4$ be divisible by $4a + 11$. The quotient has to be 1 and this is clearly impossible.

34. Let p be an odd prime. Prove that

$$1^{p-2} + 2^{p-2} + 3^{p-2} + \cdots + \left(\frac{p-1}{2}\right)^{p-2} \equiv \frac{2-2^p}{p} \pmod p.$$

Soln. All the congruences are taken mod p. For each $0 < i < p$, we have $i^{p-1} \equiv 1$. Thus $i^{p-2} \equiv \frac{1}{i}$. So

$$1^{p-2} + 2^{p-2} + 3^{p-2} + \cdots + \left(\frac{p-1}{2}\right)^{p-2} \equiv \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{(p-1)/2}.$$

On the other hand

$$\frac{2-2^p}{p} = \frac{2-(1+1)^p}{p} = \frac{1}{p}\sum_{i=1}^{p-1} -\binom{p}{i} \equiv \sum_{i=1}^{p-1}(-1)^i \frac{1}{i} \equiv -\sum_{i=1}^{p-1} \frac{1}{i} + 2\sum_{i=1}^{p-1} \frac{1}{2i} \equiv \sum_{i=1}^{(p-1)/2} \frac{1}{i}.$$

The last congruence follows because $\frac{1}{i} + \frac{1}{p-i} \equiv 0$.

Soln 2. First, for each $i = 1, 2, \cdots, \frac{p-1}{2}$,

$$\frac{2i}{p}\binom{p}{2i} = \frac{(p-1)(p-2)\cdots(p-(2i-1))}{(2i-1)!} \equiv \frac{(-1)(-2)\cdots(-(2i-1))}{(2i-1)!} \equiv -1 \pmod{p}.$$

Hence

$$\begin{aligned}\sum_{i=1}^{(p-1)/2} i^{p-2} &\equiv -\sum_{i=1}^{(p-1)/2} i^{p-2}\frac{2i}{p}\binom{p}{2i}\\ &\equiv -\frac{2}{p}\sum_{i=1}^{(p-1)/2} i^{p-1}\binom{p}{2i}\\ &\equiv -\frac{2}{p}\sum_{i=1}^{(p-1)/2} \binom{p}{2i} \qquad \pmod{p} \quad \text{by Fermat's Theorem.}\end{aligned}$$

The last summation counts the even-sized nonempty subsets of a p-element set, of which there are $2^{p-1} - 1$.

35. (Ukrainian MO 97) (10th grade) Let $d(n)$ denote the greatest odd divisor of the natural number n. Define the function $f : \mathbb{N} \to \mathbb{N}$ by $f(2n-1) = 2^n$, $f(2n) = n + 2n/d(n)$ for all $n \in \mathbb{N}$. Find all k such that $f(f(\ldots(1)\ldots)) = 1997$ where f is iterated k times.

Soln. Let $a_1 = 1$ and $a_{n+1} = f(a_n)$ for $n \geq 1$. Then $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and so on. After a few more terms, it is easy to notice that if $a_m = 2^j$, then $a_{m+j+1} = 2^{j+1}$. This can be proved as follows. Let $a_m = 2^j$. Then

$$\begin{aligned}a_{m+1} &= 2^{j-1} + 2^j = 3 \cdot 2^{j-1};\\ a_{m+2} &= 3 \cdot 2^{j-2} + 2^{j-1} = 5 \cdot 2^{j-2};\end{aligned}$$

Thus we can ofrmulate the following induction hypothesis:

$$a_{m+i} = (2i+1)2^{j-i}, \quad i = 0, 1, \ldots, j.$$

$$a_{m+i+1} = (2i+1)2^{j-i-1} + 2^{j-i} = (2(i+1)+1)2^{j-i-1}.$$

Thus the result follows by induction. Since $a_1 = 2^0$, we can find write down a formula for a_n as follows. If $n = (0+1+2+\cdots+p)+1+q$, where $p \geq 0$, $0 \leq q \leq p$, then $a_n = (2q+1)2^{p-q}$. If $(2q+1)2^{p-q} = 1997$, then $p = q = 998$. Thus $a_{499500} = 1997$. Thus $k = 499499$.

36. (Byelorrussian MO 95) Given three real numbers such that the sum of any two of them is not equal to 1, prove that there are two numbers x and y such that $xy/(x+y-1)$ does not belong to the interval $(0,1)$.

Soln. *Official solution*: Let a, b, c be the given numbers. Suppose that each of the numbers

$$A = \frac{ab}{a+b-1}, \quad B = \frac{ac}{a+c-1}, \quad C = \frac{bc}{b+c-1}.$$

belongs to $(0,1)$. Then $A > 0$, $B > 0$, $C > 0$ and

$$ABC = \frac{a^2b^2c^2}{(a+b-1)(a+c-1)(b+c-1)} > 0.$$

Hence

$$D = (a+b-1)(a+c-1)(b+c-1) > 0. \qquad (*)$$

On the other hand, $A-1 < 0$, $B-1 < 0$, $C-1 < 0$. Thus $(A-1)(B-1)(C-1) < 0$. It is easy to verify that

$$A-1 = \frac{(a-1)(b-1)}{a+b-1}, \quad B-1 = \frac{(a-1)(c-1)}{a+c-1}, \quad C-1 = \frac{(b-1)(b-1)}{b+c-1}.$$

Consequently,

$$(A-1)(B-1)(C-1) = \frac{(a-1)^2(b-1)^2(c-1)^2}{(a+b-1)(a+c-1)(b+c-1)} < 0$$

contradicting $(*)$. This proves the statement.

Solutions for 24-11-2001

52. Determine all $\alpha \in \mathbb{R}$ such that there exists a nonconstant function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(\alpha(x+y)) = f(x) + f(y).$$

Soln. For $\alpha = 1$, $f(x) = x$ satisfies the functional equation. For $\alpha \neq 1$, let $y = \frac{\alpha x}{1-\alpha}$, but this implies $f(y) = f(x) + f(y)$, hence $f(x) = 0$.

53. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying

(a) For every $n \in \mathbb{N}, f(n + f(n)) = f(n)$.

(b) For some $n_0 \in \mathbb{N}, f(n_0) = 1$.

Show that $f(n) = 1$ for all $n \in \mathbb{N}$.

Soln. Note that $f(n_0 + 1) = f(n_0 + f(n_0)) = f(n_0) = 1$. Hence $f(n) = 1$ for all $n \geq n_0$. Let $S = \{n \in \mathbb{N} | f(n) \neq 1\}$. If $S = \emptyset$, then we are done. So suppose $S \neq \emptyset$, let $M = \text{sup}S = \text{max}S$, since S is a finite set. Then $f(M + f(M)) = f(M) \neq 1$. Hence $M + f(M) \in S$, but $M + f(M) > M$, contradicting the maximality of M. Hence $S = \emptyset$, $f(n) = 1$ for all $n \in \mathbb{N}$

54. Suppose that $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(1) = 1$ and for all n,

(a) $3f(n)f(2n + 1) = f(2n)(1 + 3f(n))$,

(b) $f(2n) < 6f(n)$.

Find all (k, m) such that $f(k) + f(m) = 2001$.

Soln. Note that $\gcd(3f(n), 1 + 3f(n)) = 1$, hence $3f(n)|f(2n)$. But $f(2n) < 6f(n)$, hence $f(2n) = 3f(n)$, $f(2n + 1) = 3f(n) + 1$.Claim: If $n = (b_1b_2 \cdots b_n)_2$, then $f(n) = (b_1b_2 \cdots b_n)_3$.This is a simple exercise in induction (induct on n, consider n odd and n even separately) and is left to the reader. Then to find integers k, m such that $f(k) + f(m) = 2001$, let us first write 2001 in its tenary representation. Now $2001 = 2202010_3$. We note that the tenary representations of $f(k), f(m)$ can only contain 0 and 1's, hence there are also no carry-overs from the addition. Hence the possible pairs of k, m are only $(1101000_2, 1101010_2)$ and $(1101010_2, 1101000_2)$, i.e. (104, 106) and (106, 104).

55. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all $x, y \in \mathbb{R}$,

$$f(x^3 + y^3) = (x + y)((f(x))^2 - f(x)f(y) + (f(y))^2).$$

Prove that for all $x \in \mathbb{R}, f(2001x) = 2001f(x)$.

Soln. Let $x = y = 0$ and we have $f(0) = 0$. Setting $y = 0$, we have $f(x^3) = x(f(x))^2$, or equivalently, $f(x) = x^{1/3}f(x^{1/3})^2$. In particular, x and $f(x)$ have the same sign.Let S be the set $S = \{a > 0 : f(ax) = af(x) \forall x \in \Re\}$.Clearly $1 \in S$, and we will show $a^{1/3} \in S$ whenever $a \in S$. In fact, $axf(x)^2 = af(x^3) = f(ax^3) = f((a^{1/3}x)^3) = a^{1/3}xf(a^{1/3}x)^2$. Hence $(a^{1/3}f(x))^2 = f(a^{1/3}x)^2$. Since x and $f(x)$ have the same sign, we conclude that $f(a^{1/3}x) = a^{1/3}f(x)$.Now we show that if $a, b \in S$, then $a + b \in S$. Indeed, we have

$$\begin{aligned}
f((a + b)x) &= f((a^{1/3}x^{1/3})^3 + (b^{1/3}x^{1/3})^3) \\
&= (a^{1/3} + b^{1/3})x^{1/3}(f(a^{1/3}x^{1/3})^2 - f(a^{1/3}x^{1/3})f(b^{1/3}x^{1/3}) + f(b^{1/3}x^{1/3})^2) \\
&= (a^{1/3} + b^{1/3})(a^{2/3} - a^{1/3}b^{1/3} + b^{2/3})x^{1/3}f(x^{1/3})^2 \\
&= (a + b)f(x).
\end{aligned}$$

Hence in particular, we have $f(2001x) = 2001f(x)$.

56. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the property that for all $n \in \mathbb{N}$,

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(f(n))}{f(n+1)}.$$

Soln. Let $n = 1$ and we have $f(f(1))f(1) = 1$; hence $f(1) = 1$. Replacing the equality for n into one for $n + 1$ we obtain

$$\frac{f(f(n))}{f(n+1)} + \frac{1}{f(n+1)f(n+2)} = \frac{f(f(n+1))}{f(n+2)}.$$

This is equivalent to

$$f(f(n))f(n+2) + 1 = f(f(n+1))f(n+1).$$

Note that $f(n+1) = 1$ implies $f(f(n+1)) = 1$, hence $f(f(n))f(n+2) = 0$, which is impossible. Hence $f(n) > 1$ for $n > 1$.We use induction to show that $f(f(n)) < f(n+1)$. The inequality is true for $n = 1$, since $f(2) > 1 = f(f(1))$. Also, if $f(n+1) > f(f(n))$, then $f(n+1) \geq f(f(n)) + 1$. Hence

$$f(f(n))f(n+2) + 1 \geq f(f(n+1))f(f(n)) + f(f(n+1)).$$

Since $n+1 > 1$, $f(n+1) > 1$ and thus $f(f(n+1)) > 1$, which implies that $f(n+2) > f(f(n+1))$.The problem is now reduced to finding all functions f such that $f(f(n)) < f(n+1)$. f has a unique minimum at $n = 1$, for if $n > 1$, we have $f(n) > f(f(n-1))$. By the same reasoning, we see that the second smallest value is $f(2)$, etc. Hence

$$f(1) < f(2) < f(3) < \cdots.$$

Now since $f(1) \geq 1$, we have $f(n) \geq n$. Now suppose for some natural number n, we have $f(n) > n$. Then $f(n) \geq n + 1$. Since f is increasing, $f(f(n)) \geq f(n+1)$, a contradiction. Hence $f(n) = n$ for all $n \in \mathbb{N}$.

57. Define $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{Z}$ such that

(a) $f(x, x) = x$,

(b) $f(x, y) = f(y, x)$,

(c) $f(x, y) = f(x, x + y)$,

(d) $g(2001) = 2002$,

(e) $g(xy) = g(x) + g(y) + mg(f(x,y))$.

Determine all integers m for which g exists.

Soln. We claim that $f(x,y) = \gcd(x,y)$. This is a simple exercise in induction. (induct on the sum $x+y$).Now let $a = b$ and we have $g(a^2) = (m+2)g(a)$. Applying this again we get $g(a^4) = (m+2)g(a^2) = (m+2)^2 g(a)$.On the other hand,

$$\begin{aligned} g(a^4) &= g(a) + g(a^3) + mg(a) \\ &= (m+1)g(a) + g(a^3) \\ &= (m+1)g(a) + g(a) + g(a^2) + mg(a) \\ &= (2m+2)g(a) + g(a^2) = (3m+4)g(a). \end{aligned}$$

Now let $a = 2001$, and we have $(m+2)^2 = 3m+4$ and hence $m = 0, -1$. For $m = 0$, an example is given by

$$g(p_1^{\alpha_1} \cdots p_n^{\alpha_n}) = \alpha_1 h(p_1) + \cdots + \alpha_n h(p_n),$$

where k is a prime factor of 2001, $h(3) = 2002$ and $h(p) = 0$ for all primes $p \neq 3$. For $m = -1$, an example is given by

$$g(p_1^{\alpha_1} \cdots p_n^{\alpha_n}) = h(p_1) + \cdots + h(p_n).$$

It is easy to check that these functions do indeed satisfy the given requirements.

Solutions for 8-12-2001

63. Suppose $\{a_i\}_{i=n}^m$ is a finite sequence of integers, and there is a prime p and some k, $n \le k \le m$ such that $p | a_k$ but $p \not| a_j$, $n \le j \le m, j \neq k$. Prove that $\sum_{i=n}^m \frac{1}{a_i}$ cannot be an integer.

Soln. Suppose $\sum_{i=n}^m \frac{1}{a_i} = L$ is an integer. Then after finding common denominator and multiplying both sides by $a_n \cdots a_m$, we have

$$La_n a_{n+1} \cdots a_m = a_{n+1}a_{n+2} \cdots a_m + \cdots + a_n \cdots \widehat{a}_k \cdots a_m + \cdots + a_{n+1}a_{n+2} \cdots a_{m-1}.$$

But p divides the left hand side and all except the kth term of the right hand side. This contradiction shows that $\sum_{i=n}^m \frac{1}{a_i}$ cannot be an integer.

64. Prove that for any choice of m and n, $m, n > 1$, $\sum_{i=n}^m \frac{1}{i}$ cannot be a positive integer.

Soln. Clearly, it is meaningful to consider the cases where $m > n$. If $m \le 2n$, then $\sum_{i=n}^m \frac{1}{i} < \sum_{i=n}^m \frac{1}{n} = 1$. If $m > 2n$, then there is a prime between m and $\lfloor \frac{m}{2} \rfloor$, which we call p. From question 1, we can conclude that $\sum_{i=n}^m \frac{1}{i}$ can never be an integer.

65. There is a tournament where 10 teams take part, and each pair of teams plays against each other once and only once. Define a cycle $\{A, B, C\}$ to be such that team A beats team B, B beats C and C beats A. Two cycles $\{A, B, C\}$ and $\{C, A, B\}$ are considered the same. Find the largest possible number of cycles after all the teams have played against each other.

Soln. We label the teams $T_1, T_2, ..., T_{10}$. Let the number of times T_i wins be w_i and the number of losses be l_i. Instead of looking at the cycles as it is, we first look at the total number of 3-combinations of teams and the number of 3-combinations which cannot be rearranged to form a cycle. The total number of 3-combinations is ${}^{10}C_3 = 120$. Next, we look at the 3-combinations that can be made to form cycles and those which cannot (from here on, we call these non-cycles). The non-cycles consists of a single team beating the other two teams and a single team losing to the other 2 teams, and if a 3-combination consists of a single team beating the other two teams or a single team losing to the other 2 teams, it is a non-cycle. We can then count the non-cycles in a different manner. By focusing on this property of non-cycles, we can deduce that the number of non-cycles is $\sum_{i=1}^{10} \binom{w_i}{2} + \binom{l_i}{2}$. Clearly, we have $w_i + l_i = 9$, and rearranging the formula gives $\sum_{i=1}^{10} \binom{w_i}{2} + \binom{l_i}{2} = \sum_{i=1}^{10} \left(w_i^2 - 9w_i + 36\right)$. And the number of non-cycles and the number of cycles sum up to 120. We aim to minimise the number of non-cycles. This can be achieved if $w_i = 4$ or 5, and is certainly attainable. (The answer to this question is 40.) The description of such a case is:

(i) If T_a and T_b are such a and b have the same parity, and $a > b$, then T_a beats T_b.

(ii) If T_a and T_b are such a and b have different parity, and $a > b$, then T_a loses to T_b.

66. Consider the recursions $x_{n+1} = 2x_n + 3y_n$, $y_{n+1} = x_n + 2y_n$ with $x_1 = 2$, $y_1 = 1$. Show that for each integer $n \geq 1$, there is a positive integer K_n such that

$$x_{2n+1} = 2\left(K_n^2 + (K_n + 1)^2\right).$$

Soln. Now, after some rearrangement, we have

$$x_{2n+1} = 2\left(K_n^2 + (K_n + 1)^2\right) = (2K_n + 1)^2 + 1.$$

Therefore, it suffices to show that $x_{2n+1} - 1$ is an odd perfect square. Expressing x_{n+2} and y_{n+2} in terms of x_n and y_n, we have $x_{n+2} = 7x_n + 12y_n$ and $y_{n+2} = 4x_n + 7x_n$. As we try to express x_{n-2} in terms of x_n and y_n, we have $x_{n-2} = 7x_n - 12y_n$. Hence $x_{n+2} = 14x_n - x_{n-2}$.

We define $a_1 = 1$ and $a_3 = 5$ and $a_{n+4} = 4a_{n+2} - a_n$. We can prove by induction that $x_n - a_n^2 = 1$ for all odd integers n. Then we can proceed to prove by induction that a_n are all odd for all odd n. Therefore, $x_n - 1 = a_n^2$ and we are done.

To elaborate certain details. With $x_{n+2} = 14x_n - x_{n-2}$, $x_1 = 2$ and $x_3 = 26$, we can solve for x_n to get:

$$\begin{aligned}
x_{2n+1} &= \frac{1}{2}\left(2+\sqrt{3}\right)\left(7+4\sqrt{3}\right)^n + \frac{1}{2}\left(2-\sqrt{3}\right)\left(7-4\sqrt{3}\right)^n \\
&= \frac{1}{2}\left(2+\sqrt{3}\right)\left(2+\sqrt{3}\right)^{2n} + \frac{1}{2}\left(2-\sqrt{3}\right)\left(2-\sqrt{3}\right)^{2n} \\
&= \frac{1}{4}\left(4+2\sqrt{3}\right)\left(2+\sqrt{3}\right)^{2n} + \frac{1}{4}\left(4-2\sqrt{3}\right)\left(2-\sqrt{3}\right)^{2n} - 1 + 1 \\
&= \left(\frac{1}{2}\left(1+\sqrt{3}\right)\right)^2\left(2+\sqrt{3}\right)^{2n} + \left(\frac{1}{2}\left(1-\sqrt{3}\right)\right)^2\left(2-\sqrt{3}\right)^{2n} - 1 + 1 \\
&= \left(\frac{1}{2}\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)^n\right)^2 + \left(\frac{1}{2}\left(1-\sqrt{3}\right)\left(2-\sqrt{3}\right)^n\right)^2 - 1 + 1 \\
&= \left(\frac{1}{2}\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)^n + \frac{1}{2}\left(1-\sqrt{3}\right)\left(2-\sqrt{3}\right)^n\right)^2 + 1
\end{aligned}$$

67. Suppose that $ABCD$ is a tetrahedron and its four altitudes AA_1, BB_1, CC_1, DD_1 intersect at the point H. Let A_2, B_2, C_2 be points on AA_1, BB_1, CC_1 respectively such that $AA_2 : A_2A_1 = BB_2 : B_2B_1 = CC_2 : C_2C_1 = 2 : 1$. Show that the points H, A_2, B_2, C_2, D_1 are on a sphere.

Soln. Let G be the centroid of the triangle ABC. Prove that $\angle HA_2G = \angle HB_2G = \angle HC_2G = \angle HD_1G = 90^o$. Thus, The centre of the sphere that we are looking for is at the midpoint of H and G.