Training problems 10 April 2003

14. Let A be a set with 8 elements. Find the maximum number of distinct 3-element subsets of A such that the intersection of any two of them is not a 2-element set.

Solution. Let G be a graph with vertex set A. Two vertices are adjacent if and only if they belong to the same 3-element subset. Thus 3 vertices that belong to the same 3-element subset will form a 3-cycle (C_3). The question asks for the maximum number of C_3s that G can contain subject to the condition that every 2 of these C_3s do not have a common edge. (The graph may, however, contain other C_3. But these will share an edge with the C_3s we want.) We may further assume that every edge belongs to some C_3.

Since at most 3 C_3 can share a common vertex, every vertex if od degree at most 6. Thus G has at most $(6 \times 8)/2 = 24$ edges and so at most 8 C_3s.

An example with 8 sets is:

$$123, 145, 167, 246, 278, 348, 357, 568.$$

2nd solution. This is not easier than the first but it introduces a very useful idea of an incidence matrix.

Let $B_1, \ldots, B_n$ be 3-element subsets of A such that $|B_i \cap B_j| \neq 2$. Form an incidence matrix with rows indexed by $B_1, \ldots, B_n$ and with columns indexed by the elements $a_1, \ldots, a_8$ of A. An entry (B_i, a_j) is 1 if $a_j \in B_i$ and is 0 otherwise. Then there are 3 ones in every row. Call a pair of ones in the same column a 1-pair. The given condition states that there is at most 1 1-pair in every pair of rows. Suppose there is a column, say column 1, that has 4 ones, say in the first 4 rows. Then in the submatrix formed by the first 4 rows and the last 7 columns, there are 8 ones. Thus there is at least one 1-pair. Hence some pair of rows has two 1-pairs, a contradiction.

Thus every column has at most 3 ones. Counting the total number of ones in the incidence matrix in 2 different ways, we conclude that $n \leq 8$.

It's not hard to get an example with 8 sets. Thus the answer is 8.

15. Find all primes p for which $p(2^{p-1} - 1)$ is the kth power of a positive integer for some $k > 1$.

Solution. Let $p(2^{p-1} - 1) = x^k$ for some positive integers x, k. It's clear that $p \neq 2$. Thus $p = 2q + 1$. Write $x = py$. Then $(2^q + 1)(2^q - 1) = p^{k-1}y^k$. Since at least one of $2^q + 1$ and $2^q - 1$ is the kth power of an integer since they are coprime.

Case (1): $2^q - 1 = z^k$. Then $2^q = z^k + 1$. If k is even, then $z^k + 1$ is not divisible by 4. Hence $q = 1, p = 3$ and $p(2^{p-1} - 1) = 3^2$.

If $k = 2\ell + 1$, i.e., odd, then

$$2^q = (z + 1)(z^{2\ell} - z^{2\ell-1} + \cdots - z + 1) \quad \text{i.e} \quad z + 1 = 2^\alpha, 0 \leq \alpha < q.$$

On the other hand,

$$2^q = (2^\alpha - 1)^{2\ell+1} + 1 = A2^{2\alpha} + 2^\alpha(2\ell + 1), \quad A \text{ is an integer}$$

The last equality contradicts with $\alpha < q$.

Case (2): $2^q + 1 = z^k$. Then $2^q = z^k - 1$. If k is odd we get a contradiction as before.

If $k = 2\ell$, then $(z^\ell - 1)(z^\ell + 1) = 2^q$ and since $\gcd(z^\ell - 1, z^\ell + 1) = 2$, we have $z^\ell - 1 = 2$, i.e., $q = 3$, $p = 7$, $p(2^{p-1} - 1) = 21^2$.

Thus the answers are $p = 3, 7$.

16. Let k be a given real number. Find all functions $f : (0, \infty) \to (0, \infty)$ such that the following equality holds for all positive real number x:

$$kx^2 f(1/x) + f(x) = \frac{x}{x+1}.$$

Solution. Divide by x we get

$$kxf(1/x) + \frac{1}{x} f(x) = \frac{1}{x+1}$$

Replace x by $1/x$ we get

$$\frac{k}{x} f(x) + xf(1/x) = \frac{x}{1+x}.$$

Solve the system of equations with unknowns $f(x)$ and $f(1/x)$, we get

$$\frac{(1-k^2)f(x)}{x} = \frac{1-kx}{x+1}.$$

If $k \neq \pm 1$, then there is a unique solution

$$f(x) = \frac{x}{x+1} \frac{1-kx}{1-k^2}$$

It's easy to see that this expression satisfies the given functional equation.

If $k = \pm 1$, there is no solution. For $f(x) > 0$ for all positive x, we also need $-1 < k < 0$. Thus such a function exists only when $-1 < k < 0$. In this case the unique solution is $f(x) = \frac{x}{x+1} \frac{1-kx}{1-k^2}$.

17. Let n be an integer, $n \geq 3$. Let $a_1, \ldots, a_n$ be real numbers, where $2 \leq a_i \leq 3$ for $i = 1, \ldots, n$. if $s = a_1 + \cdots + a_n$. prove that

$$\frac{a_1^2 + a_2^2 - a_3^2}{a_1 + a_2 - a_3} + \frac{a_2^2 + a_3^2 - a_4^2}{a_2 + a_3 - a_4} + \cdots + \frac{a_n^2 + a_1^2 - a_2^2}{a_n + a_1 - a_2} \leq 2s - 2n.$$

Solution. Write

$$A_i = \frac{a_i^2 + a_{i+1}^2 - a_{i+2}^2}{a_i + a_{i+1} - a_{i+2}} = a_i + a_{i+1} + a_{i+2} - \frac{2a_i a_{i+1}}{a_i + a_{i+1} - a_{i+2}}.$$

Since $(a_i - 2)(a_{i+1} - 2) \geq 0$, $-2a_i a_{i+1} \leq -4(a_i + a_{i+1} - 2)$ and

$$A_i \leq a_i + a_{i+1} + a_{i+2} - 4\left(1 + \frac{a_{i+2} - 2}{a_i + a_{i+1} - a_{i+2}}\right).$$

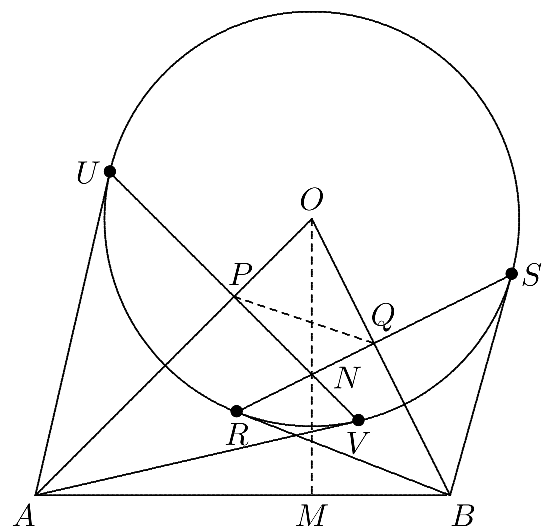
Since $1 = 2 + 2 - 3 \leq a_i + a_{i+1} - a_{i+2} \leq 3 + 3 - 2 = 4$,

$$A_i \leq a_i + a_{i+1} + a_{i+2} - 4\left(1 + \frac{a_{i+2} - 2}{4}\right) = a_i + a_{i+1} - 2.$$

Hence $\sum A_i \leq 2s - 2n$.

18. Two chords UV and RS of a circle $\mathcal{C}$ centred at O intersect at the point N. Suppose AB is a line segment outside the circle $\mathcal{C}$ such that AU, AV, BR and BS are tangent to the circle $\mathcal{C}$ at U, V, R and S respectively. Prove that ON is perpendicular to AB.

Solution.



Join ON and extend it to meet AB at M. Let OA intersect UV at P and OB intersect RS at Q. Join PQ. Then $\angle OPN = \angle OQN = 90°$. Hence, O, P, N and Q are concyclic. As $OP \cdot OA = OU^2 = OS^2 = OQ \cdot OB$, we have A, B, Q and P are concyclic. Therefore, $\angle OAM = \angle OQP = \angle ONP$. This shows that P, A, M and N are concyclic. Hence, $\angle AMO = \angle OPN = 90°$.

(*2nd solution by Colin Tan*) Extend BN to W such that $\angle NWS = \angle BSN$ ($= \angle BRN$). This is possible as $\angle BNS > \angle BRN$. Thus BS is tangent to circumcircle of SNW and $SWRB$ cyclic. This gives the relations $BN \cdot BW = BS^2 = OB^2 - OR^2$ and $BN \cdot NW = SN \cdot NR$ so $BN^2 = BN \cdot BW - BN \cdot NW = OB^2 - OR^2 - SN \cdot NR$. Get a similar expression for AN^2, and this would give $BN^2 - AN^2 = OB^2 - OA^2$ which implies that ON is perpendicular to AB. (*Compare the proof of this with question 12. Also the same proof using projective geometry as in question 12 can be applied here.*)

19. Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AC - BD| \leq |AB - CD|.$$

When does equality hold?

Solution. Let E and F be the midpoints of the diagonals AC and BD. In every quadrilateral the following relation due to Euler holds:

$$AC^2 + BD^2 + 4EF^2 = AB^2 + BC^2 + CD^2 + DA^2.$$

Since $ABCD$ is a cyclic quadrilateral, we have Ptolemeus identity

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

Hence,

$$(AC - BD)^2 + 4EF^2 = (AB - CD)^2 + (AD - BC)^2.$$

Let us prove that $4EF^2 \geq (AD - BC)^2$. This will implies the stated inequality. Let M be the midpoint of AB. In the triangle MEF, we have $AD = 2MF$, $BC = 2ME$, and from triangle's inequality, $EF \geq |ME - MF|$, hence $2EF \geq |BC - AD|$ and $4EF^2 \geq (AD - BC)^2$.

The equality holds if and only if the points M, E, F are collinear, which happens if and only if AB is parallel to CD, that is $ABCD$ is either an isosceles trapezium or a rectangle.

20. Let Γ be a convex polygon with 2000 sides and P an interior point which does not lie on any diagonal of Γ. Prove that P is in the interior of an even number of triangles formed using the vertices of Γ.

Solution. First observe that if P lies in a quadrilateral, then it is contained in the interiors of two triangles. Next, if a triangle $\triangle$ contains P, then any quadrilateral containing $\triangle$ also contains P. As each triangle in Γ is contained in 1997 quadrilaterals, the point $P \in \triangle$ is contained in 1997 quadrilaterals. Let m be the number of quadrilaterals containing P and n the number of triangles containing P. Then $2m = 1997n$. Hence, n must be even. Here, we are counting the number of pairs $(\triangle, Q)$, where P lies in the triangle $\triangle$ which is inside the quadrilateral Q.