Singapore International Mathematical Olympiad
Training Problems

18 January 2003

1. Let $M$ be a point on the segment $AB$. Squares $AMCD$ and $MBEF$ are erected on the same side of $AB$ with $F$ lying on $MC$. The circumcircles of $AMCD$ and $MBEF$ meet at a second point $N$. Prove that $N$ is the intersection of the lines $AF$ and $BC$.

2. Let $DM$ be the diameter of the incircle of a triangle $ABC$ where $D$ is the point at which the incircle touches the side $AC$. The extension of $BM$ meets $AC$ at $K$. Prove that $AK = CD$.

3. Tangents $PA$ and $PB$ are drawn from a point $P$ outside a circle $Γ$. A line through $P$ intersects $AB$ at $S$ and $Γ$ at $Q$ and $R$. Prove that $PS$ is the harmonic mean of $PR$ and $PQ$.

4. (IMO 1981) Three circles of equal radius have a common point $O$ and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point $O$. 

1. Let $M$ be a point on the segment $AB$. Squares $AMCD$ and $MBEF$ are erected on the same side of $AB$ with $F$ lying on $MC$. The circumcircles of $AMCD$ and $MBEF$ meet at a second point $N$. Prove that $N$ is the intersection of the lines $AF$ and $BC$.

Solution

Let $AF$ intersect $BC$ at $N'$. We wish to show that $N = N'$. As $\triangle AMF$ is congruent to $\triangle CMB$, we have $\angle AN'C = 90^\circ$ so that $N'$ lies on the circle with $AC$ as diameter. That is $N'$ lies on the circumcircle of $AMCD$. Similarly, $N'$ lies on the circumcircle of $MBEF$. Thus $N = N'$.

2. Let $DM$ be the diameter of the incircle of a triangle $ABC$ where $D$ is the point at which the incircle touches the side $AC$. The extension of $BM$ meets $AC$ at $K$. Prove that $AK = CD$.

Solution
Consider a homothety centered at $B$ carrying the incircle to the excircle. The diameter $MD$ of the incircle is mapped to the diameter $M'D'$ of the excircle. Since $MD$ is perpendicular to $AC$, $M'D'$ is also perpendicular to $AC$. Therefore $M'$ must be the point $K$. That is the excircle touches $AC$ at $K$. Therefore, $AK = (a + b - c)/2 = CD$.

3. Tangents $PA$ and $PB$ are drawn from a point $P$ outside a circle $\Gamma$. A line through $P$ intersects $AB$ at $S$ and $\Gamma$ at $Q$ and $R$. Prove that $PS$ is the harmonic mean of $PR$ and $PQ$.

**Solution** Since $\triangle APQ$ is similar to $\triangle RPA$, we have $PQ/PA = AQ/AR$. Also $\triangle BPQ$ is similar to $\triangle RPB$, we have $PR/PB = RB/QB$. Dividing the second equation by the first equation and using the fact that $PA = PB$, we obtain $PR/PQ = (RB/AQ) \cdot (AR/QB) = (RS/AS) \cdot (AS/QS) = SR/SQ$. This shows that the ratio that $S$ divides $QR$ internally is the same as the ratio that $P$ divides $QR$ externally. This determines the position of $S$ on the segment $QR$.

![Diagram of tangents PA, PB, and PS](image)

Thus

$$SR = QR \cdot \frac{PR}{PR + PQ}, \quad SQ = QR \cdot \frac{PQ}{PR + PQ}.$$  

Also

$$PS = PR + RS = PR + \frac{QR \cdot PR}{PR + PQ} = PR + \frac{(PQ - PR) \cdot PR}{PR + PQ} = \frac{2PR \cdot PQ}{PR + PQ}. $$

That is $PS$ is the harmonic mean of $PR$ and $PQ$.

(Second Solution by Colin Tan) Let $M$ be the midpoint of $QR$. Then to prove $PS = 2PR \cdot PQ/(PR + PQ)$ is equivalent to prove that $PS \cdot PM = PR \cdot PQ$. Or equivalently, $PS \cdot PM = PB^2$, since $PR \cdot PQ = PB^2$. Therefore, we have to show that $PB$ is tangent to the circumcircle of $\triangle SMB$. Let $O$ be the centre of $\Gamma$. Then $O, M, P, B$ are concyclic and $OP$ is perpendicular to $AB$. Hence, $\angle PBA = \angle POB = \angle SMB$. Therefore, $PB$ is tangent to the circumcircle of $\triangle SMB$.

3
(Third Solution) Applying Stewart’s Theorem to \( \triangle APB \), we have

\[
(AS + SB) \cdot PS^2 + (AS + SB) \cdot AS \cdot SB = AP^2 \cdot SB + BP^2 \cdot AS.
\]

Since \( AP = BP \), we may cancel the common factor \( (AS + SB) \), thus obtaining

\[
PS^2 + AS \cdot SB - AP^2 = 0.
\]

Since \( AS \cdot SB = QS \cdot SR = (PQ - PS)(PS - PR) = (PQ + PR) \cdot PS - PS^2 - PQ \cdot PR \) and \( PA^2 = PQ \cdot PS \), we have \( (PQ + PR) \cdot PS = 2PQ \cdot PR \). Thus, \( PS \) is the harmonic mean of \( PR \) and \( PQ \).

4. (IMO 1981) Three circles of equal radius have a common point \( O \) and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point \( O \).

Solution
Let $A', B', C'$ be the centres of the circles inside $\triangle ABC$. As $AA', BB', CC'$ are angle bisectors, they meet at the incenter $I$ of triangle $ABC$. $I$ is also the incenter of the triangle $A'B'C'$. The circles are of the same radii. Thus $A'$ and $B'$ are of equal distance from $AB$ so that $AB$ is parallel to $A'B'$. Similarly, $BC$ is parallel to $B'C'$ and $A'C'$ is parallel to $AC$. That is $\triangle ABC$ is similar to $\triangle A'B'C'$. Consider a homothety centred at $I$ sending $A'$ to $A$, $B'$ to $B$ and $C'$ to $C$. Thus the circumcentre $O$ of $\triangle A'B'C'$ is mapped to the circumcentre $C$ of $\triangle ABC$ under this homothety. Therefore, $I, C, O$ are collinear.