Singapore International Mathematical Olympiad Training Problems

25 January 2003

1. In a quadrilateral $ACGE$, H is the intersection of AG and CE, the lines AE and CG meet at I and the lines AC and EG meet at D. Let B be the intersection of the line IH and AC. Prove that $AB/BC = AD/DC$, or equivalently, DB is the harmonic mean of DA and DC.

2. The extensions of the chords QR and $Q'R'$ of a circle Γ intersect at a point P outside Γ. Tangents PA and PA' are drawn from P to Γ. Prove that A, X, A' are collinear where X is the intersection of QR' and $Q'R$.

3. In a quadrilateral $ABCD$, E is a point on CD, BE intersects AC at F and the extension of DF meets BC at G. Suppose that AC bisects $\angle BAD$. Prove that $\angle GAC = \angle EAC$.

4. (Crux 2333) Points D and E are on the sides AC and AB of $\triangle ABC$. Suppose F and G are points of BC and ED, respectively, such that $BF : FC = EG : GD = BE : CD$. Prove that GF is parallel to the angle bisector of $\angle BAC$.

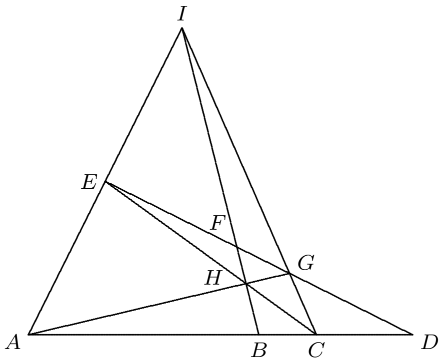
5. (Balkan Math Olympiad 2002) Let O be the center of the circle through the points A, B, C and let D be the midpoint of AB. Let E be the centroid of triangle ACD. Prove that the line CD is perpendicular to the line OE if and only if $AB = AC$.

1. In a quadrilateral $ACGE$, H is the intersection of AG and CE, the lines AE and CG meet at I and the lines AC and EG meet at D. Let B be the intersection of the line IH and AC. Prove that $AB/BC = AD/DC$, or equivalently, DB is the harmonic mean of DA and DC.

Solution Apply Menelaus' Theorem to $\triangle ACI, \triangle AEC$ and $\triangle CEI$ with transversals EGD, IHB and AHG respectively. We have

$$\frac{CD}{DA}\frac{AE}{EI}\frac{IG}{GC} = 1, \qquad \frac{AB}{BC}\frac{CH}{HE}\frac{EI}{IA} = 1, \qquad \frac{CG}{GI}\frac{IA}{AE}\frac{EH}{HC} = 1.$$

The result is obtained by multiplying these three equations together.



Alternatively, by Ceva's Theorem applied to $\triangle ACI$, we have

$$\frac{IE}{EA}\frac{AB}{BC}\frac{CG}{GI} = 1.$$

Next by Menelaus' Theorem applied to $\triangle ACI$ with transversal EGD, we have

$$\frac{AD}{DC}\frac{CG}{GI}\frac{IE}{EA} = 1.$$

Thus, $AB/BC = AD/DC$.

2. The extensions of the chords QR and $Q'R'$ of a circle Γ intersect at a point P outside Γ. Tangents PA and PA' are drawn from P to Γ. Prove that A, X, A' are collinear where X is the intersection of QR' and $Q'R$.

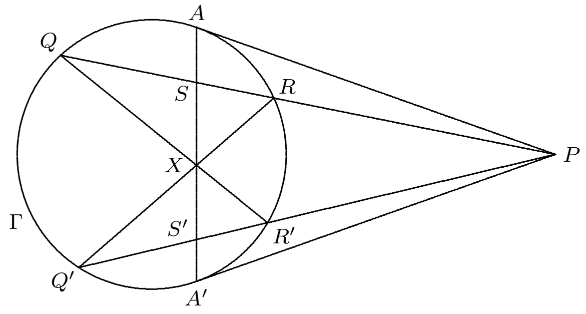
Solution Let AA' intersect PQ at S and PQ' at S'. We wish to prove that S, X, S' are collinear. We know that PS is the harmonic mean of PQ and PR or equivalently $PQ : PR = SQ : SR$. (See problem 3 in the training problems on 18 January 2003.) Thus

$$PS = \frac{2PQ \cdot PR}{PR + PQ} = \frac{2PQ}{1 + \frac{PQ}{PR}} = \frac{2PQ}{1 + \frac{SQ}{SR}} = \frac{2PQ \cdot SR}{QR}.$$

Therefore, $\dfrac{PS}{SR} = \dfrac{2PQ}{QR}$. Also $\dfrac{PS}{SQ} = \dfrac{PS}{SR}\dfrac{SR}{SQ} = \dfrac{2PQ}{QR}\dfrac{PR}{PQ} = \dfrac{2PR}{QR}$.

Note that all these ratios are equivalent to each other and they are just different ways to express the location of S. Similarly,

$$\frac{PS'}{S'R'} = \frac{2PQ'}{Q'R'} \quad \text{and} \quad \frac{PS'}{S'Q'} = \frac{2PR'}{Q'R'}.$$



Apply Menelaus' Theorem to $\triangle PRQ'$ and the transversal QXR'. We get

$$\frac{XR}{XQ'} = \frac{PR'}{PQ}\frac{QR}{Q'R'} = \frac{PR}{PQ'}\frac{QR}{Q'R'},$$

because $\triangle PRQ'$ is similar to $\triangle PR'Q$. This characterizes the position of X along RQ'. Now consider $\triangle PRQ'$ and the transversal SXS'. Combining the above results, we have

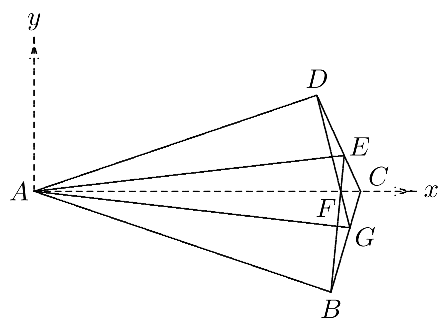
$$\frac{PS}{SR}\frac{RX}{XQ'}\frac{Q'S'}{S'P} = \frac{PS}{SR}\frac{PR}{PQ'}\frac{QR}{Q'R'}\frac{Q'S'}{S'P} = \frac{2PQ}{QR}\frac{PR}{PQ'}\frac{QR}{Q'R'}\frac{Q'R'}{2PR'} = \frac{PQ}{PQ'}\frac{PR}{PR'} = 1.$$

By Menelaus' Theorem, S, X, S' and hence A, X, A' are collinear.

3. In a quadrilateral $ABCD$, E is a point on CD, BE intersects AC at F and the extension of DF meets BC at G. Suppose that AC bisects $\angle BAD$. Prove that $\angle GAC = \angle EAC$.

Solution Let A be the origin of a rectangular coordinates system with AC as the x-axis. Let $C = (c, 0)$, $F(f, 0)$, $D = (x_D, kx_D)$, $B = (x_B, -kx_B)$. Then the equation of the line DF is

$$x - f + \frac{f - x_D}{kx_D} y = 0 \qquad (1)$$



The equation of the line BC is

$$x - c + \frac{c - x_B}{-kx_B} y = 0 \qquad (2)$$

By taking $c \times (1) - f \times (2)$, we have

$$(c-f)x + \frac{1}{k}\left[cd\left(\frac{1}{x_D}+\frac{1}{x_B}\right)-(c+f)\right]y = 0 \tag{3}$$

This is the equation of a line passing through A (as there is no constant term) and the intersection of DF and BC. Hence, it is the equation of the line AG. Similarly, the equation of the line AE is given by

$$(c-f)x - \frac{1}{k}\left[cd\left(\frac{1}{x_D}+\frac{1}{x_B}\right)-(c+f)\right]y = 0 \tag{4}$$

From (3) and (4), we see that the slopes of the lines AG and AE are negative of each other. Therefore, $\angle GAC = \angle EAC$.

(Second Solution by Colin Tan and Meng Dazhe)

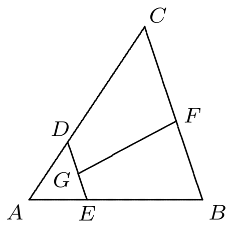
Pick a point G' on BC such that $\angle G'AC = \angle BAC$ (let this angle is a). Join BD to intersect AC at H. Let $\angle BAG' = \angle EAD$ be b.

Consider the Cevians CH, BE, DG' of $\triangle BCD$. Now $EG'/G'C = [ABG']/[AG'C] = (AB \sin a)/(AC \sin b)$. Similarly, we find that $CE/ED = (AC \sin b)/(AD \sin a)$ and $DH/HB = (AD \sin(a+b))/(AB \sin(a+b))$. Thus, $(BG'/G'C)(CE/ED)(DH/HB) = 1$.

So by Ceva's theorem, we conclude that BE, CH and DG' are concurrent (at F). Hence, G', F, D are collinear implying that $G' = G$. Therefore $\angle GAC = \angle EAC$ as required.

4. (Crux 2333) Points D and E are on the sides AC and AB of $\triangle ABC$. Suppose F and G are points of BC and ED, respectively, such that $BF : FC = EG : GD = BE : CD$. Prove that GF is parallel to the angle bisector of $\angle BAC$.

Solution Let A be the origin of a rectangular coordinates system. For each of the points in the question, we use the small case letter in bold face to denote the position vector of that point. First we have $\mathbf{e} = p\mathbf{b}$ and $\mathbf{d} = q\mathbf{c}$ for some p, q in $(0,1)$.



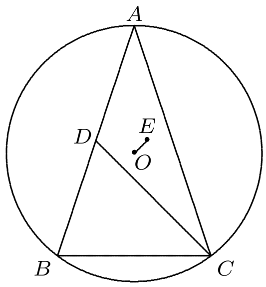
Let $t = BF/FC$. Then $\mathbf{f} = \dfrac{t\mathbf{c}+\mathbf{b}}{t+1}$ and $\mathbf{g} = \dfrac{t\mathbf{d}+\mathbf{e}}{t+1} = \dfrac{tq\mathbf{c}+p\mathbf{b}}{t+1}$. Since $BE = tCD$, so $(1-p)|\mathbf{b}| = t(1-q)|\mathbf{c}|$. Thus,

$$\mathbf{f}-\mathbf{g} = \frac{t(1-q)}{t+1}\mathbf{c} + \frac{1-p}{t+1}\mathbf{b} = \frac{(1-p)|\mathbf{b}|}{t+1}\left(\frac{\mathbf{c}}{|\mathbf{c}|}+\frac{\mathbf{b}}{|\mathbf{b}|}\right).$$

This is parallel to $\dfrac{\mathbf{c}}{|\mathbf{c}|}+\dfrac{\mathbf{b}}{|\mathbf{b}|}$, which is in the direction of the angle bisector of $\angle BAC$.

5. (Balkan Math Olympiad 2002) Let O be the center of the circle through the points A, B, C and let D be the midpoint of AB. Let E be the centroid of triangle ACD. Prove that the line CD is perpendicular to the line OE if and only if $AB = AC$.

Solution Set the origin at O. As in the last question, for each of the points in the question, we use the small case letter in bold face to denote the position vector of that point. Then $\mathbf{d} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and $\mathbf{e} = \frac{1}{3}(\mathbf{a} + \mathbf{c} + \mathbf{d}) = \frac{1}{6}(3\mathbf{a} + \mathbf{b} + 2\mathbf{c})$, $\mathbf{d} - \mathbf{c} = \frac{1}{2}(\mathbf{a} + \mathbf{b} - 2\mathbf{c})$.



Hence CD is perpendicular to OE if and only if $(\mathbf{a} + \mathbf{b} - 2\mathbf{c}) \cdot (3\mathbf{a} + \mathbf{b} + 2\mathbf{c}) = 0$. Since $\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{c}$, this is equivalent to $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$, which is the same as OA is perpendicular to BC, or $AB = AC$.