Training problems 27 March 2003

1. Find all integer solutions in x and y of the equation $x^3 + 27xy + 2009 = y^3$.

Solution. Let $y = x + a$. Then the equation becomes

$$(27 - 3a)x^2 + (27a - 3a^2)x - a^3 + 2009 = 0.$$

As a quadratic equation in x, its discriminant is

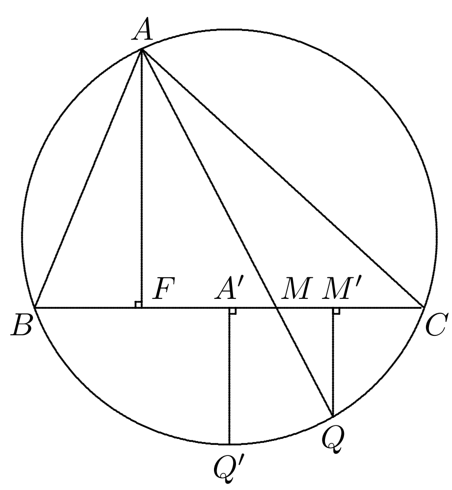
$$(27a - 3a^2)^2 - 4(27 - 3a)(-a^3 + 2009) = -3(a - 14)(a - 9)(a^2 + 41a + 574).$$

The factor $a^2 + 41a + 574$ is always positive. Therefore the equation has integer solution in x only when $a = 9$, 10, 11, 12, 13, 14. When $a = 9$, the equation becomes $-3x^2 - 30x + 1009 = 0$ which has no integer solution in x. Similarly for $a =$ 10,11,12,13, the resulting quadratic equations do not give integer solution in x. When $a = 14$, the equation becomes $-15(x+7)^2 = 0$. Thus $x = -7, y = a+x = 14-7 = 7$ is a solution to the given equation.

2. Let ABC be a given triangle, and M, N, and P be arbitrary points in the interiors of the line segments BC, CA, and AB respectively. Let the lines AM, BN, and CP intersect the circumcircle of ABC in points Q, R, and S respectively. Prove that

$$\frac{AM}{MQ} + \frac{BN}{NR} + \frac{CP}{PS} \geq 9.$$

Solution. Let F be the foot of the perpendicular from A onto BC. If Q' is the midpoint of the arc BC and A' is the midpoint of BC, it is clear that $AM/MQ = AF/M'Q \geq AF/A'Q'$. Therefore, the minimum value of $\frac{AM}{MQ} + \frac{BN}{NR} + \frac{CP}{PS}$ will be obtained uniquely where Q, R, S are the midpoints of the arcs BC, CA and AB respectively. Thus, we will henceforth assume that Q, R, S are positioned so that AQ, BR, QS are the angle bisectors of angles A, B, C respectively.



We know that the angle-bisector AQ cuts the side BC of length a in the ratio $c : b$ so $BM = ca/(b + c)$ and $MC = ba/(b + c)$. Therefore,

$$\frac{AM}{MQ} = \frac{AM^2}{AM \cdot MQ} = \frac{AM^2}{BM \cdot MC} = \frac{AM^2(b + c)^2}{a^2bc}. - - - (*)$$

From Stewart's Theorem,

$$c^2MC + b^2BM - aBM \cdot MC - aAM^2 = 0.$$

It follows that

$$AM^2 = bc - \frac{ca^2b}{(b+c)^2}.$$

Substituting this into (*), we have

$$\frac{AM}{MQ} = \left(\frac{b+c}{a}\right)^2 - 1.$$

Similarly,

$$\frac{BN}{NR} = \left(\frac{a+c}{b}\right)^2 - 1, \quad \frac{CP}{PS} = \left(\frac{a+b}{c}\right)^2 - 1.$$

Thus we have

$$\frac{AM}{MQ} + \frac{BN}{NR} + \frac{CP}{PS} \geq \left(\frac{b+c}{a}\right)^2 + \left(\frac{a+c}{b}\right)^2 + \left(\frac{a+b}{c}\right)^2 - 3.$$

By the convexity of $f(x) = x^2$, we have

$$\left(\frac{b+c}{a}\right)^2 + \left(\frac{a+c}{b}\right)^2 + \left(\frac{a+b}{c}\right)^2 \geq \frac{1}{3}\left(\frac{b+c}{a} + \frac{a+c}{b} + \frac{a+b}{c}\right)^2.$$

Also, using $AM \geq GM$, and the inequality $x + 1/x \geq 2$ for $x > 0$, we get

$$\frac{AM}{MQ} + \frac{BN}{NR} + \frac{CP}{PS} \geq \frac{1}{3}\left[\left(\frac{b}{a} + \frac{a}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) + \left(\frac{c}{b} + \frac{b}{c}\right)\right]^2 \geq 9.$$

Equality holds if and only if $a = b = c$; that is to say, if and only if triangle ABC is equilateral and $M.N.P$ are the midpoints of the sides.

3. Does there exist a convex pentagon, all of whose vertices are lattice points in the xy-plane, with no lattice point in the interior? (A point in the xy-plane is called a lattice point if it has integer coordinates.)

Solution. The answer is No. A convex lattice pentagon must have an interior lattice point. To see this, note that every lattice point (x, y) belongs to one of the four classes K_{00}, K_{01}, K_{10} and K_{11}, where the index pair ij is determined by taking $i \equiv x$ and $j \equiv y$ modulo 2. A convex lattice pentagon has five vertices, so two of them, say P and Q belong to the same class, which implies that their midpoint R is also a lattice point. If P and Q are endpoints of a diagonal of the pentagon, then R is an interior lattice point. If P and Q are the endpoints of an edge, say edge AB of the pentagon $ABCDE$, we continue by considering the convex lattice pentagon $ARCDE$ and so on. This case cannot continue indefinitely, because if so, there would be an infinite sequence of distinct lattice points within a finite region of the coordinate plane, which is not true.

4. A piece of cardboard in the shape of a square is to be cut into n acute- angled triangles. Find the smallest n for which this can be done. Show at least one way to do it this minimum n.

Solution. Suppose the square has been cut into n acute-angled triangles. Form a graph whose vertices are the vertices of the square and the vertices of the triangles. Any two consecutive vertices on a side of a triangle or on a side of a square are joined by an edge. There are no other edges. There are three types of vertices:

(a) Vertices of the square: These are of degree at least 3.

(b) Vertices which are in the interior of a side of a triangle or the square. These are of degree at least 4. Denote the number of vertices of these type by b

(c) Vertices of the triangles which do not lie in the interior of a side. These are of degree at least 5. Denote the number of vertices of these type by c

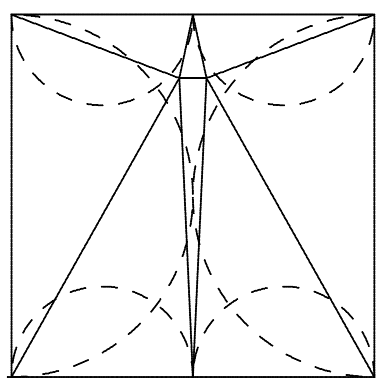
Let m be the number of edges. Then

$$2m \geq 4 \times 3 + 4b + 5c.$$

But $2m = 4 + 3n + b$. Euler's formula gives

$$(4 + b + c) + (n + 1) - m = 3, \quad \text{or} \quad m = b + c + n + 3.$$

Substitute this into the inequality, we get $c \geq 2$. Each vertex of type (c) is associated with at least 5 triangles and the triangles associated to 2 vertices can have a overlap of at most 2. Thus we get $n \geq 8$. See the picture below for a construction which gives $n = 8$.



5. (**Prize Problem**) Let x be a positive rational number. Prove that there exist a unique set of integers $a_1, a_2 \ldots, a_k$, with $0 \leq a_n \leq n - 1$ for $n > 1$ such that

$$x = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_k}{k!}.$$

Show also that there exist a set of integers, $10^6 < b_1 < b_2 < \cdots b_m$ such that

$$x = \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_m}.$$

Solution. First note that x can be written in the form $m/n!$, where m, n are positive integers. We shall prove by induction on n that

$$\frac{m}{n!} = a_1 + \frac{a_2}{2!} + \cdots + \frac{a_n}{n!}$$

where $0 \le a_i \le i-1$ for $i \ge 2$. For $n = 1$, $a_1 = m$, and $a_i = 0$ for $i > 1$ works. Now assume that it holds for some n. Let

$$m = (n+1)q + r, \qquad 0 \le r < n+1$$

Then

$$\frac{m}{(n+1)!} = \frac{q}{n!} + \frac{r}{(n+1)!}.$$

The result then follows by applying the induction hypothesis on $q/n!$ and putting $r = a_{k+1}$. Next we prove uniqueness. Suppose

$$x = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_k}{k!} = b_1 + \frac{b_2}{2!} + \frac{b_3}{3!} + \cdots + \frac{b_k}{k!}.$$

Then

$$a_k \equiv b_k \equiv x(k!) \pmod k$$

and it follows that $a_k = b_k$. By considering $x - (a_k/k!)$, we can show that $a_{k-1} = b_{k-1}$. Thus uniqueness follows.

To prove the second part we first note that if $a_i > 0$, then $a_i \mid i!$ and thus $a_i/i!$ is the reciprocal of a positive integer. Moreover, $(i-1)! < i!/a_i \le i!$. Thus all the positive integers are different.

Now if $x < 1$, then $a_1 = 0$ and so x can be expressed as the sum of reciprocals of different positive integers.

Now we consider the general case. Let x be any positive rational number and m be an integer $> 10^6$ such that $m > 1/x$ and n be the largest integer $\ge m$ such that

$$x \ge \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n}.$$

Let

$$y = x - \frac{1}{m} - \frac{1}{m+1} - \cdots - \frac{1}{n}.$$

Then $0 \le y < 1/(n+1)$. Thus y can be written as a sum of reciprocals of different positive integers:

$$y = \frac{1}{q_1} + \cdots + \frac{1}{q_j}.$$

Since $1/q_i < y < 1/(n+1)$, $q_i > (n+1)$. Thus

$$x = \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n} + \frac{1}{q_1} + \cdots + \frac{1}{q_j}$$

as required.

6. Given a segment AB of length 1, define the set M of points as follows: $A, B \in M$ and if $X, Y \in M$, then M contains the point Z in the segment XY for which $YZ = 3XZ$. Prove that M does not contain the midpoint of AB.

Solution. Represent A by 0 and B by 1 on the number line. Denote by M_n the set of points of the segment AB obtained from A, B by not more than n iterations. It can be proved by induction that M_n consists of all points in $[0, 1]$ represented by $3k/4^n$ and $(3k-2)/4^n$, where k is an integer. Thus M consists of numbers of the form $3k/4^n$ and $(3k-2)/4^n$. To prove our assertion, we need to show that $1/2$ cannot be expressed in this form. Suppose $1/2 = 3k/4^n$, then $4^n = 6k$ which has no solution. Suppose $1/2 = (3k-2)/4^n$, then $6n = 4^n + 4$ which has no solution. Thus $1/2 \notin M$.

2nd soln (Joel): Represent each point by a coordinate in the set $[0, 1]$ with $A = 0$, $B = 1$. For any two points $x, y \in M$, the point $(3x + y)/4 \in M$. Now take decimal representations. Note that each point is a finite decimal. It's easy to see that the digit sum of the decimal representation $\equiv 0 or 1 \pmod 3$ for any point in M since it initially holds for A, B and if it holds x, y then it also holds for $(3x + y)/4$. Since midpoint of AB has decimal representation of 0.5, it is not in M.

7. (**Prize Problem**) Let $a_1, a_2, \ldots, a_n$, $n \geq 1$, be real numbers ≥ 1 and $A = 1 + a_1 + \cdots + a_n$. Define x_k, $0 \leq k \leq n$ by

$$x_0 = 1, \quad x_k = \frac{1}{1 + a_k x_{k-1}}, \ 1 \leq k \leq n.$$

Prove that

$$x_1 + x_2 + \cdots + x_n > \frac{n^2 A}{n^2 + A^2}.$$

(Hint: Let $y_k = 1/x_k$)

Solution. Let $y_k = 1/x_k$. We then have $y_k = 1 + \frac{a_k}{y_{k-1}}$. From $y_{k-1} \geq 1$, $a_k \geq 1$ we obtain

$$\left(\frac{1}{y_{k-1}} - 1\right)(a_k - 1) \leq 0 \quad \Leftrightarrow \quad 1 + \frac{a_k}{y_{k-1}} \leq a_k + \frac{1}{y_{k-1}}.$$

So $y_k = 1 + \frac{a_k}{y_{k-1}} \leq a_k + \frac{1}{y_{k-1}}$. We have

$$\sum_{k=1}^{n} y_k \leq \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} \frac{1}{y_{k-1}} = \sum_{k=1}^{n} a_k + \frac{1}{y_0} + \sum_{k=1}^{n-1} \frac{1}{y_k} = A + \sum_{k=1}^{n-1} \frac{1}{y_k} < A + \sum_{k=1}^{n} \frac{1}{y_k}.$$

Let $t = \sum_{k=1}^{n} 1/y_k$. Then $\sum_{k=1}^{n} y_k \geq n^2/t$. So for $t > 0$,

$$\begin{aligned} n^2 t < A + t \quad &\Leftrightarrow \quad t^2 + At - n^2 \geq 0 \\ &\Leftrightarrow \quad t > \frac{-A + \sqrt{A^2 + 4n^2}}{2} = \frac{2n^2}{A + \sqrt{A^2 + 4n^2}} \\ &\geq \frac{2n^2}{A + A + (2n^2/A)} = \frac{n^2 A}{n^2 + A^2}. \end{aligned}$$