Singapore International Mathematical Olympiad 2008
Senior Team Training

Take Home Test Solutions

1. Show that the equation $15x^2 - 7y^2 = 9$ has no solution in integers.

If the equation has a solution in integer, then

$$\begin{aligned} & 15x^2 - 7y^2 = 9 \\ \Rightarrow\quad & -y^2 \equiv 0 \pmod 3 \\ \Rightarrow\quad & y \equiv 0 \pmod 3. \end{aligned}$$

Hence $y = 3y_1$ for some integer y_1. This implies that

$$\begin{aligned} & 15x^2 - 7(3y_1)^2 = 9 \\ \Rightarrow\quad & 5x^2 - 21y_1^2 = 3 \\ \Rightarrow\quad & 2x^2 \equiv 0 \pmod 3 \\ \Rightarrow\quad & x \equiv 0 \pmod 3. \end{aligned}$$

Hence $x = 3x_1$ for some integer x_1. This implies that

$$\begin{aligned} & 15(3x_1)^2 - 7(3y_1)^2 = 9 \\ \Rightarrow\quad & 15x_1^2 - 7y_1^2 = 1 \\ \Rightarrow\quad & -y_1^2 \equiv 1 \pmod 3 \\ \Rightarrow\quad & y_1^2 \equiv 2 \pmod 3. \end{aligned}$$

The last congruence is impossible.
Hence the given equation has no solution in integers.

2. Let n and k be positive integers. Prove that

$$(n^4 - 1)(n^3 - n^2 + n - 1)^k + (n + 1)n^{4k-1}$$

is divisible by $n^5 + 1$.

We prove by induction on k:
When $k = 1$, we have

$$\begin{aligned} & (n^4 - 1)(n^3 - n^2 + n - 1) + (n + 1)n^3 \\ = \quad & n^7 - n^6 + n^5 + n^2 - n + 1 \\ = \quad & n^2(n^5 + 1) - n(n^5 + 1) + n^5 + 1 \\ = \quad & (n^5 + 1)(n^2 - n + 1). \end{aligned}$$

So the statement is true for $k = 1$. Now assume that the statement is true for k and consider the case $k+1$, we have

$$\begin{aligned}
& (n^4-1)(n^3-n^2+n-1)^{k+1}+(n+1)n^{4(k+1)-1} \\
= & (n^4-1)(n^3-n^2+n-1)^k(n^3-n^2+n-1)+(n+1)n^{4k-1}n^4 \\
= & [(n^4-1)(n^3-n^2+n-1)^k(n^3-n^2+n-1)+(n+1)n^{4k-1}(n^3-n^2+n-1)] \\
+ & [(n+1)n^{4k-1}n^4-(n+1)n^{4k-1}(n^3-n^2+n-1)] \\
= & (n^3-n^2+n-1)[(n^4-1)(n^3-n^2+n-1)^k+(n+1)n^{4k-1}] \\
+ & n^{4k-1}(n+1)(n^4-n^3+n^2-n+1) \\
= & (n^3-n^2+n-1)[(n^4-1)(n^3-n^2+n-1)^k+(n+1)n^{4k-1}]+n^{4k-1}(n^5+1).
\end{aligned}$$

This is divisible by n^5+1 by induction hypothesis. Hence the statement is true for $k+1$. This completes the induction.

3. Let

$$\begin{aligned}
f(x) &= (x+1)^p(x-3)^q \\
&= x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n,
\end{aligned}$$

where p and q are positive integers.

(1) Given that $a_1 = a_2$, prove that $3n$ is a perfect square.
(2) (b) Prove that there exist infinitely many pairs (p, q) of positive integers p and q such that the equality $a_1 = a_2$ is valid for the polynomial $p(x)$.

(Belarussian 2003)
(a) Since

$$\begin{aligned}
(x+1)^p &= x^p+px^{p-1}+\frac{p(p-1)}{2}x^{p-2}+\cdots \\
(x-3)^q &= x^q-3qx^{q-1}+\frac{9q(q-1)}{2}x^{q-2}+\cdots,
\end{aligned}$$

we have $n = p+q$, $a_1 = p-3q$, $a_2 = \frac{9q^2-9q+p^2-p-6pq}{2}$. Therefore

$$\begin{aligned}
& a_1 = a_2 \\
\Leftrightarrow & 2p-6q = 9q^2-9q+p^2-p-6pq \\
\Leftrightarrow & (3q-p)^2 = 3(p+q).
\end{aligned}$$

Since $n = p+q$, we are done.

(b) This is equivalent to showing that the equation $(3q-p)^2 = 3(p+q)$ has an infinite family of solutions in positive integers. Treating this as a quadratic equation in p,

$$9p^2 - (6q+3)p + (9q^2 - 3q) = 0$$

we have

$$p = 6q + 3 \pm \sqrt{\frac{48q+9}{2}}.$$

Thus $48q+9$ is an odd square. Hence

$$48q + 9 = (2k+1)^2, \text{ or } q = \frac{k^2+k-2}{12}$$

Let $k = 12t+1$, then $q = 12t^2 + 3t$ and $p = 36t^2 - 3t$, where $t \in \mathbb{N}$ is the required infinite family.

4. Show that if $m < n$, then $2^{2^m} + 1$ divides $2^{2^n} - 1$. Hence deduce that $2^{2^m} + 1$ and $2^{2^n} + 1$ are relatively prime. Conclude that there are infinitely many primes.

If $m < n$ then $n = m + k$ for some integer $k \geq 1$, so we have

$$2^{2^n} - 1 = 2^{2^m 2^k} - 1 = (2^{2^m} + 1)(2^{2^m(2^k-1)} - 2^{2^m(2^k-2)} + - \cdots + 2^{2^m} - 1).$$

Hence $2^{2^m} + 1$ divides $2^{2^n} - 1$.

Let $d = \gcd(2^{2^m} + 1, 2^{2^n} + 1)$. By above, we have

$$\begin{aligned}
& 2^{2^n} + 1 = (2^{2^n} - 1) + 2 = l(2^{2^m} + 1) + 2 \text{ for some integer } l \\
\Rightarrow\ & d|2 \quad (\text{since } d|(2^{2^m}+1) \text{ and } d|(2^{2^n}+1)) \\
\Rightarrow\ & d = 1 \text{ or } d = 2 \\
\Rightarrow\ & d = 1 \quad (\text{since } 2^{2^m} + 1 \text{ is odd})
\end{aligned}$$

Thus $\gcd(2^{2^m}+1, 2^{2^n}+1) = 1$, i.e. $2^{2^m}+1$ and $2^{2^n}+1$ are relatively prime. For any positive integer n, let p_n be a prime divisor of $2^{2^n}+1$. For any m, n with $n \neq m$, we have $p_n \neq p_m$ since $2^{2^n}+1$ and $2^{2^m}+1$ are relatively prime. Hence $\{p_1, p_2, p_3 \ldots\}$ is an infinite set of primes.

5. Let x, y, z be positive numbers so that $xyz = 1$. Prove that

$$x + y + z \geq \sqrt[3]{\frac{z}{x}} + \sqrt[3]{\frac{x}{y}} + \sqrt[3]{\frac{y}{z}}.$$

By AM-GM inequality,

$$y + z + z \geq 3\sqrt[3]{yz^2} = 3\sqrt[3]{\frac{z}{x}},$$

$$x + x + z \geq 3\sqrt[3]{x^2 z} = 3\sqrt[3]{\frac{x}{y}},$$

and

$$x + y + y \geq 3\sqrt[3]{xy^2} = 3\sqrt[3]{\frac{y}{z}}.$$

Summing the three inequalities gives the desired result.

6. Let $p_1, p_2, \ldots, p_n$ $(n \geq 2)$ be any rearrangement of $1, 2, \ldots n$. Show that

$$\frac{1}{p_1 + p_2} + \frac{1}{p_2 + p_3} + \cdots + \frac{1}{p_{n-1} + p_n} > \frac{n-1}{n+2}.$$

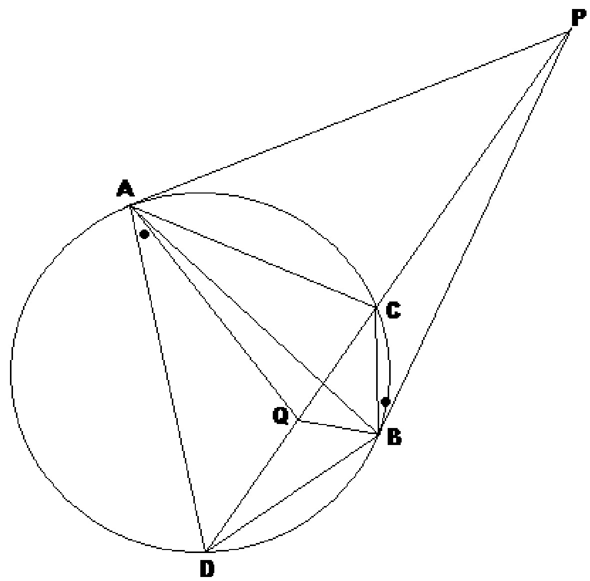
Since $AM \geq HM$,

$$\frac{1}{n-1}[(p_1 + p_2) + (p_2 + p_3) + \cdots + (p_{n-1} + p_n)]$$
$$\geq \left\{\frac{1}{n-1}\left[\frac{1}{p_1 + p_2} + \cdots + \frac{1}{p_{n-1} + p_n}\right]\right\}^{-1}.$$

Thus

$$\frac{1}{p_1 + p_2} + \cdots + \frac{1}{p_{n-1} + p_n}$$
$$\geq \frac{(n-1)^2}{2(p_1 + \cdots + p_n) - p_1 - p_n}$$
$$\geq \frac{(n-1)^2}{n(n+1) - 3} = \frac{(n-1)^2}{(n-1)(n+2) - 1}$$
$$> \frac{(n-1)^2}{(n-1)(n+2)} = \frac{n-1}{n+2}.$$

7. From a point P outside a circle, tangent lines PA and PB are drawn with A and B on the circle. A third line PCD meets the circle at C and D, with C lying in between P and D. A point Q is chosen on the chord CD so that $\angle DAQ = \angle PBC$. Show that $\angle DBQ = \angle PAC$.



Since

$$\angle ABC = \angle ADQ$$
$$\angle BAC = \angle PBC = \angle DAQ,$$

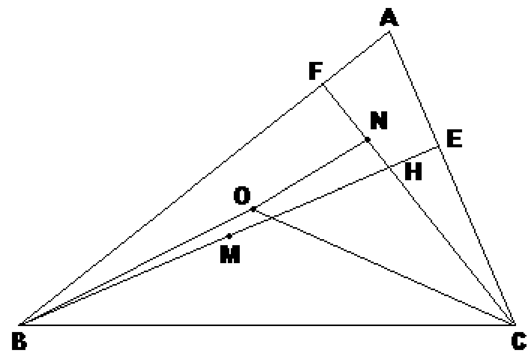
$\triangle ADQ \sim \triangle ABC$. Thus $BC \cdot AD = AB \cdot DQ$. Also, $\triangle PCA \sim \triangle PAD$. Hence $\frac{PC}{PA} = \frac{AC}{AD}$. Similarly, $\frac{PC}{PB} = \frac{BC}{BD}$. But $PA = PB$. So we have $\frac{AC}{AD} = \frac{BC}{BD}$, and thus $AC \cdot BD = BC \cdot AD = AB \cdot DQ$. By Ptolemy's Theorem,

$$AC \cdot BD + BC \cdot AD = AB \cdot CD.$$

Therefore, $AB \cdot CD = 2AB \cdot DQ$, or $DQ = \frac{1}{2}CD$. So Q is the midpoint of CD. Now $\frac{AD}{AB} = \frac{DQ}{BC} = \frac{CQ}{BC}$ and $\angle BCQ = \angle BAD$. It follows that $\triangle CBQ \sim \triangle ABD$. Hence $\angle CBQ = \angle ABD$. Thus $\angle DBQ = \angle ABC = \angle PAC$.

8. In triangle ABC, $\angle A = 60^\circ$ and $AB > AC$. The altitudes BE and CF intersect at H. Points M and N are chosen on the segments BH and HF so that $BM = CN$. If O is the circumcircle of ABC, find the ratio

$$\frac{MH + NH}{OH}.$$



Note that $\angle BOC = 2\angle A = 120^\circ$ and $\angle BHC = 180^\circ - \angle A = 120^\circ$ (since A, E, H, F are concyclic). Hence B, O, H, C are concyclic. Thus $\angle OBH = \angle OCH$. As $BO = CO$ and $BM = CN$ as well, $\triangle OBM$ is congruent to $\triangle OCN$. Hence $OM = ON$ and $\angle BMO = \angle CNO$. It follows that O, M, H, N are concyclic. Therefore, $\angle NOM = \angle NHE = 120^\circ$. Also,

$$\angle ONM = \angle OHM = \angle OHB = \angle OCB = 30^\circ.$$

Thus

$$\frac{MN}{OM} = \frac{\sin 120^\circ}{\sin 30^\circ} = \sqrt{3}.$$

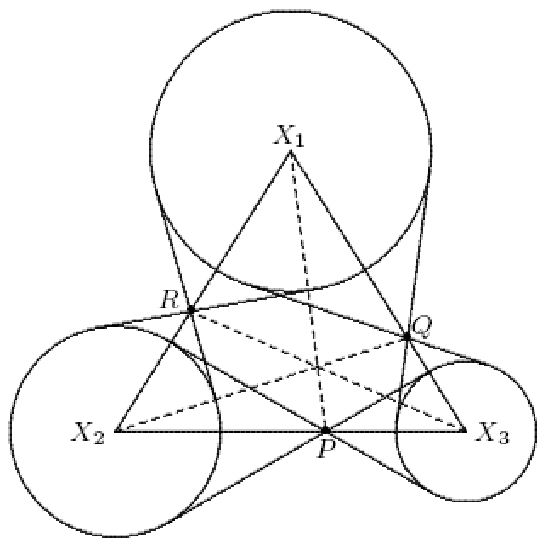
Finally, from Ptolemy's Theorem, we have

$$MH \cdot ON + NH \cdot OM = OH \cdot MN.$$

Therefore,

$$\frac{MH + NH}{OH} = \frac{MN}{OM} = \sqrt{3}.$$

9. On the plane, there are 3 mutually and externally disjoint circles Γ_1, Γ_2 and Γ_3 centred at X_1, X_2 and X_3 respectively. The two internal common tangents of Γ_2 and Γ_3, (Γ_3 and Γ_1, Γ_1 and Γ_2) meet at P, (Q, R respectively). Prove that X_1P, X_2Q and X_3Z are concurrent.

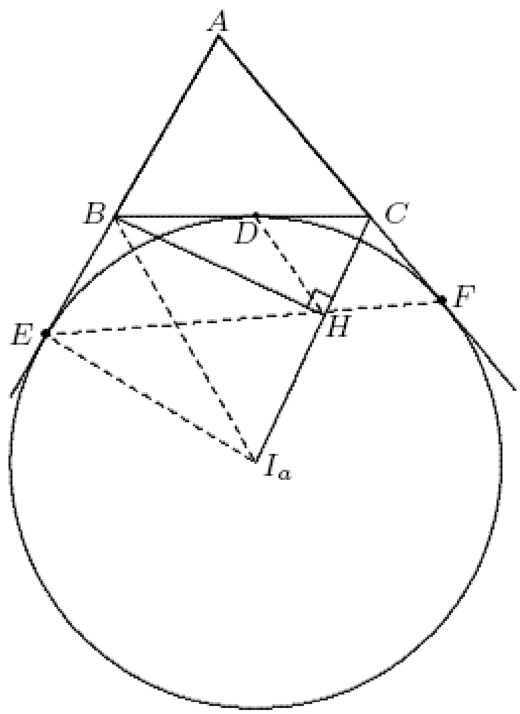


Let the radii of Γ_1, Γ_2 and Γ_3 be r_1, r_2 and r_3 respectively. Then $X_1R : RX_2 := r_1 : r_2$, $X_2P : PX_3 := r_2 : r_3$ and $X_3Q : QX_1 := r_3 : r_1$. Thus

$$\frac{X_1R}{RX_2} \cdot \frac{X_2P}{PX_3} \cdot \frac{X_3Q}{QX_1} = 1.$$

By the converse of Ceva's Theorem, X_1P, X_2Q and X_3Z are concurrent.

10. The excircle centred at I_a with respect to $\angle A$ of $\triangle ABC$ touches the sides AB, BC and AC or their extensions at E, D and F respectively. Let H be the foot of the perpendicular from B onto I_aC. Prove that E, H, F are collinear.



Join EH, FH, DH, EI_a, BI_a. Then E, I_a, H, B are concylic. Also D, B, E, I_a are concylic. Thus B, E, I_a, H, D all lie on a circle with diameter BI_a. As $BD = BE$, we have $\angle BHD = \angle BHE$. On the other hand, $\triangle DHC$ is congruent to $\triangle CHF$ so that $\angle DHC = \angle CHF$. As $\angle BHC = 90°$, we have $\angle DHE + \angle DHF = 2\angle BHC = 180°$. Therefore, E, H, F are collinear.