Singapore International Mathematical Olympiad 2009
Senior Team Training

Take Home Test Solutions

1. For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation has precisely three 1s.

(1) Prove that, for each positive integer m, there exists at least one positive integer k such that $f(k) = m$.

(2) Determine all positive integers m for which there exists exactly one k with $f(k) = m$.

Let $g(n) = 1$ if the binary expansion of n has exactly 3 '1's and $g(n) = 0$ otherwise.

$$f(k+1) - f(k) = g(2k+2) + g(2k+1) - g(k+1)$$

Since $g(2k+2) = g(k+1)$, $f(k+1) - f(k)$ is either 1 or 0 depending on whether $g(2k+1)$ is 1 or 0. Note that the binary expansions of numbers between 2^n and 2^{n+1} have $n+1$ digits. Thus among these numbers there are $\binom{n}{2}$ which have 3 '1's in the binary representation. Thus $f(2^n) = \binom{n}{2}$. Since $f(1) = 0$ and the image of f is $\mathbb{Z}^+$. This proves (a).

Let m be any positive integer for which there is only one k with $f(k) = m$. Then

$$f(k+1) - f(k) = 1 = f(k) - f(k-1).$$

This means $g(2k+1) = g(2k-1) = 1$. This means the binary expansion of k has exactly 2 '1's and ends in 10. Thus $k = 2^n + 2$ for some n. Clearly, there are infinitely many such k. From $2^n + 3$ to 2^{n+1} there are $\binom{n}{2}$ numbers whose binary representation has exactly 3 '1's. From $2^{n+1} + 1$ to $2^{n+1} + 4$ there is only 1 such number. Thus $f(2^n + 2) = 1 + \binom{n}{2}$.

2. Determine all pairs (n, p) of positive integers such that p is prime, $n \leq 2p$, and $(p-1)^n + 1$ is divisible by n^{p-1}.

Clearly $(1, p)$ and $(2, 2)$ are solutions and for other solutions we have $p \geq 3$. Now assume that $n \geq 2$ and $p \geq 3$. Since $(p-1)^n + 1$ is odd and is divisible by n^{p-1}, n must be odd. Thus $n < 2p$. Let q be the smallest prime divisor of n. From $q \mid (p-1)^n + 1$, we have

$$(p-1)^n \equiv -1 \pmod{q} \quad \text{and} \quad \gcd(q, p-1) = 1.$$

But $\gcd(n, q-1) = 1$ (from the choice of q), there exist integers u and v such that $un + v(q-1) = 1$, whence

$$p - 1 \equiv (p-1)^{un}(p-1)^{v(q-1)} \equiv (-1)^u 1^v \equiv -1 \pmod{q},$$

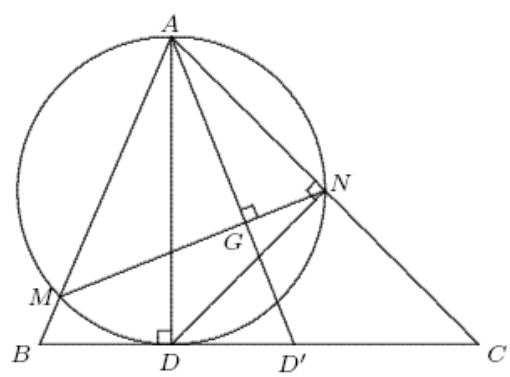
because u must be odd. This shows $q \mid p$ and therefore $q = p$. Hence $n = p$. Now

$$\begin{aligned} &p^{p-1} \mid (p-1)^p + 1 \\ &\quad = p^2 \left(p^{p-2} - \binom{p}{1} p^{p-3} + \cdots + \binom{p}{p-3} p - \binom{p}{p-2} + 1 \right). \end{aligned}$$

Since every term in the bracket except the last is divisible by p, we have $p - 1 \leq 2$. Thus $p = 3 = n$. Indeed $(3, 3)$ is a solution.
In conclusion, the only solutions are $(1, p), (2, 2), (3, 3)$.

3. Let AD, BE and CF be the altitudes of the triangle ABC. The circle with diameter AD meets AB at M and AC at N and the line through A perpendicular to MN meets BC at D'. Similar points E' and F' are constructed on the sides AC and AB respectively. Prove that AD', BE', CF' are concurrent.

Let AD' intersect MN at G. Join DN.

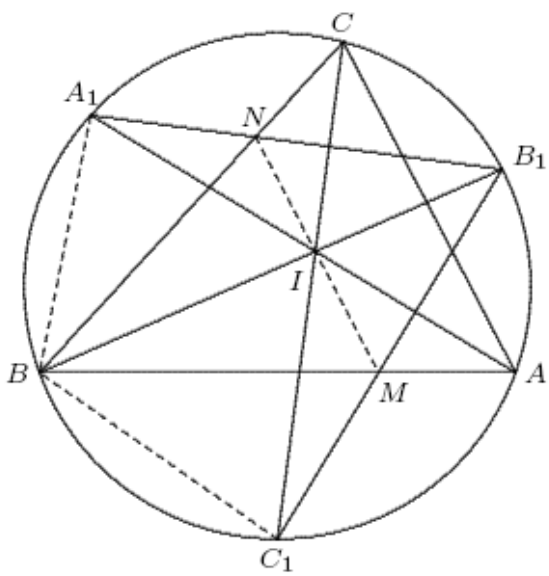


Then $\triangle AND$ is similar to $\triangle AGM$ so that $\angle BAD' = \angle MAG = \angle DAC$ and $\angle D'AC = \angle DAB$. Similarly, $\angle CBE' = \angle ABE$, $\angle E'BA = \angle EBC$ and $\angle ACF' = \angle BCF$, $\angle F'CB = \angle FCA$. Thus

$$\frac{\sin\angle BAD'}{\sin\angle D'AC} \cdot \frac{\sin\angle ACF'}{\sin\angle F'CB} \cdot \frac{\sin\angle CBE'}{\sin\angle E'BA} = \frac{\sin\angle CAD}{\sin\angle DAB} \cdot \frac{\sin\angle ABE}{\sin\angle EBC} \cdot \frac{\sin\angle BCF}{\sin\angle FCA} = 1,$$

since the three altitudes concur at the orthocentre. Thus by Ceva's theorem, AD', BE', CF' are concurrent.

4. In an acute triangle ABC, the bisector of $\angle A, \angle B$ and $\angle C$ intersect the circumcircle again at points A_1, B_1 and C_1, respectively. Let M be the point of intersection of AB and B_1C_1, and let N be the point of intersection of BC and A_1B_1. Prove that MN passes through the incentre of $\triangle ABC$.



Join A_1B, BC_1, NI and IM. First we have $\angle NA_1I = \angle B_1A_1A = \angle B_1BA = \angle NBI$. That means NA_1BI are concyclic. Similarly, $\angle MC_1I = \angle MBI$ so that MC_1BI are concyclic. Thus $\angle MIB = 180^\circ - \angle B_1C_1B = NA_1B$. Therefore, $\angle MIB + \angle BIN = 180^\circ$. In other words, N, I, M are collinear.

5. If $x_1, \ldots, x_n$ are positive real numbers, where $n \geq 5$, show that

$$\frac{x_1^3}{x_1^3 + x_2x_3x_4} + \frac{x_2^3}{x_2^3 + x_3x_4x_5} + \cdots + \\ + \frac{x_{n-1}^3}{x_{n-1}^3 + x_nx_1x_2} + \frac{x_n^3}{x_n^3 + x_1x_2x_3} \leq n - 1.$$

Let $y_1 = x_2x_3x_4/x_1^3, \ldots, y_n = x_1x_2x_3/x_n^3$. Then $y_1 \ldots y_n = 1$.

Claim. If $m \geq 2$ and $u_1, \ldots, u_m$ are positive numbers with $u_1 \cdots u_m = 1$, then

$$\sum_{k=1}^{m} \frac{u_k}{1 + u_k} \geq 1.$$

For $m = 2$, the claim is clear. Suppose that the claim holds for some $m \geq 2$.

If $u_1, \ldots, u_m, u_{m+1}$ are positive numbers with product 1, then

$$\sum_{k=1}^{m-1} \frac{u_k}{1+u_k} + \frac{u_m u_{m+1}}{1+u_m u_{m+1}} \geq 1.$$

Since

$$\frac{u_m}{1+u_m} + \frac{u_{m+1}}{1+u_{m+1}} \geq \frac{u_m u_{m+1}}{1+u_m u_{m+1}},$$

the claim follows by induction.

From the claim, we see that

$$\sum_{k=1}^{n} \frac{y_k}{1+y_k} \geq 1.$$

This is equivalent to

$$\sum_{k=1}^{n} \frac{1}{1+y_k} \leq n-1,$$

which is equivalent to the inequality to be shown.

6. Let a, b, c be the three sides of a triangle. Prove that

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3.$$

By Chebyshev's inequality,

$$\begin{aligned}\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} &\geq \frac{a+b+c}{3}\Big[\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c}\Big] \\ &\geq \frac{a+b+c}{3}\Big[\frac{9}{a+b+c}\Big]\end{aligned}$$

7. Arrange distinct real numbers $a_1, \ldots, a_n$ in a circular pattern in the given order in the clockwise direction. Assume that $a_1 + \cdots + a_n = 0$. Show that there exists an integer k, $1 \leq k \leq n$, so that the sum of any i $(1 \leq i \leq n)$ consecutive terms in the clockwise direction beginning with a_k is nonnegative.

Let $S_j = a_1 + \cdots + a_j$, $1 \leq j \leq n$. If $S_j \geq 0$ for all j, then $k = 1$ satisfies the requirements. Otherwise, pick k so that

$$S_{k-1} = \min\{S_1, S_2, \ldots, S_n\} < 0.$$

(If the minimum occurs at S_n, take k to be 1.) If $k \leq j \leq n$, then

$$a_k + \cdots + a_j = S_j - S_{k-1} > 0.$$

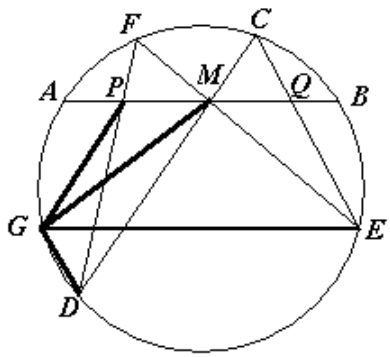
If $1 \leq j \leq k-1$, then, since $S_n = 0$ by assumption,

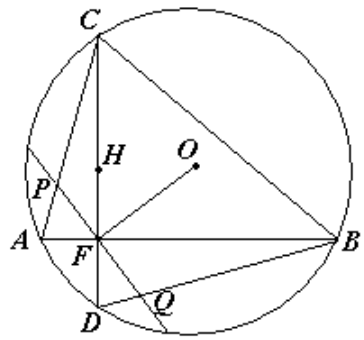
$$a_k + \cdots + a_n + a_1 + \cdots + a_j = S_n - S_{k-1} + S_j = S_j - S_{k-1} \geq 0.$$

8. In a tournament with n players, every pair of players play each other exactly once, and each match results in a win for one of the players. A player, say Player A, is awarded a Grand Prize if for any other player B, either A beats B or there is another player C so that A beats C and C beats B. Show that the tournament must have at least one Grand Prize winner. If there is exactly one Grand Prize winner, show that the winner has beaten every other player.

There is a player with the most number of wins. Let's say it's player n and she has m wins. We may assume that player n has beaten players $1, \ldots, m$ and lost to players $m+1, \ldots, n-1$. Consider player k, $m+1 \leq k \leq n-1$. She has beaten player n and she has at most m wins. So she must have lost to at least one of the players $1, \ldots, m$, whom player n beats. This shows that player n is a Grand Prize winner.
Suppose that player n is a Grand Prize Winner who has beaten players $1, \ldots, m$ and lost to players $m+1, \ldots, n-1$. Consider the mini-tournament among players $m+1, \ldots, n-1$. By the first part, there must be a mini-Grand Prize winner. Let it be player $m+1$. This means that for each player k, $m+2 \leq k \leq n$, player $m+1$ has beaten player k or beaten another player who has beaten player k. For each player k, $1 \leq k \leq m$, player $m+1$ has beaten player n, who has beaten player k. Also player $m+1$ has beaten player n. Therefore, player $m+1$ is also a Grand Prize winner. So if there is only one Grand Prize winner, she must have beaten all other players.

9. Let ABC be an acute triangle with $BC > CA$. Denote its circumcenter and orthocenter by O and H respectively. Extend CH to meet AB at F. The line through F perpendicular to OF meets CA at P. Show that $\angle FHP = \angle BAC$.





We first recall the Butterfly Theorem:

Let M be the midpoint of a chord AB of a given circle. Let C, D, E, F be points on the circle so that the chords CD and EF pass through M. Let P be the intersection of AB with DF and Q be the intersection of AB with CE. Then $PM = MQ$.

Construct chord FG parallel to AB. Since M is the midpoint of AB, the triangle GME is isosceles. Thus

$$\angle PMG = \angle MGE = \angle MEG = \angle FEG = \angle FDG = \angle PDG.$$

Hence the points P, M, D, G are concyclic. Therefore,

$$\angle PGM = \angle PDM = \angle FDM = \angle FEC = \angle MEQ.$$

Since $\angle PMG = \angle QME$ and $MG = ME$ as well, the triangles PMG and QME are congruent. So $PM = MQ$.

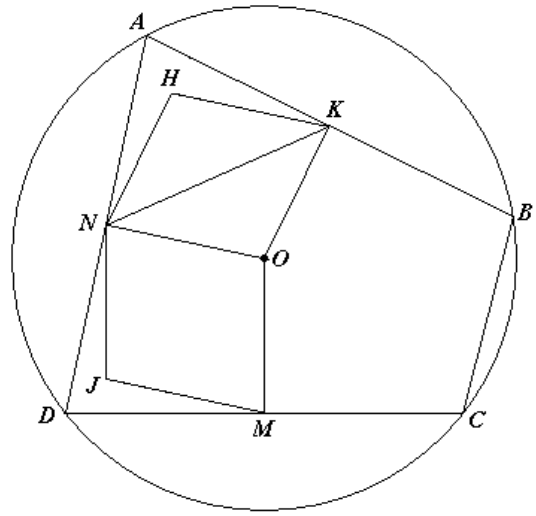
Now we return to the proof of the exercise. Extend CF to meet the circumcircle at D. Extend FP to a chord of the circle and let this chord meet DB at Q. Since OF is perpendicular to the chord, F is the midpoint of the chord. By the Butterfly Theorem, $PF = FQ$. Since H is the orthocenter, $BH \perp AC$. Hence

$$\angle DHB = \angle FHB = 90^\circ - \angle HBF = 90^\circ - \angle HBA = \angle BAC = \angle BDC = \angle BDH.$$

Thus triangle BHD is isosceles. Since $BF \perp HD$, F must be the midpoint of HD. So $FH = HD$. Obviously, $\angle PFH = \angle QFD$. So the triangles PHF and QDF are congruent. In particular, $\angle PHF = \angle FDQ = \angle CDB = \angle BAC$.

10. Let $ABCD$ be a cyclic quadrilateral and let points K, L, M, N be the midpoints of the sides AB, BC, CD and DA respectively. Show that the

orthocenters of the four triangles AKN, BKL, CLM and DMN are the vertices of a parallelogram.



Let O be the center of the circumcircle of $ABCD$. Then OK, OL, OM, ON are perpendicular to the chords AB, BC, CD and DA respectively. Let H be the orthocenter of triangle ANK. Then NH is perpendicular to AK and hence parallel to OK. Similarly, KH is parallel to ON. In the same manner, the orthocenter J of triangle DNM is the fourth vertex of the parallelogram $NOMJ$. It follows easily that HJ is parallel to KM. Similarly, if G and I are the orthocenters of triangles BLK and CLM respectively, then GI is parallel to KM. Also, HG and JI are both parallel to NL. Thus $HGIJ$ is a parallelogram.