

**49th INTERNATIONAL MATHEMATICAL OLYMPIAD**  
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*Thursday, July 17, 2008*

**Problem 4.** Find all functions  $f : (0, \infty) \rightarrow (0, \infty)$  (so,  $f$  is a function from the positive real numbers to the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers  $w, x, y, z$ , satisfying  $wx = yz$ .

**Problem 5.** Let  $n$  and  $k$  be positive integers with  $k \geq n$  and  $k - n$  an even number. Let  $2n$  lamps labelled  $1, 2, \dots, 2n$  be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequences of *steps*: at each step one of the lamps is switched (from on to off or from off to on).

Let  $N$  be the number of such sequences consisting of  $k$  steps and resulting in the state where lamps  $1$  through  $n$  are all on, and lamps  $n + 1$  through  $2n$  are all off.

Let  $M$  be the number of such sequences consisting of  $k$  steps, resulting in the state where lamps  $1$  through  $n$  are all on, and lamps  $n + 1$  through  $2n$  are all off, but where none of the lamps  $n + 1$  through  $2n$  is ever switched on.

Determine the ratio  $N/M$ .

**Problem 6.** Let  $ABCD$  be a convex quadrilateral with  $|BA| \neq |BC|$ . Denote the incircles of triangles  $ABC$  and  $ADC$  by  $\omega_1$  and  $\omega_2$  respectively. Suppose that there exists a circle  $\omega$  tangent to the ray  $BA$  beyond  $A$  and to the ray  $BC$  beyond  $C$ , which is also tangent to the lines  $AD$  and  $CD$ . Prove that the common external tangents of  $\omega_1$  and  $\omega_2$  intersect on  $\omega$ .