36th International Mathematical Olympiad

Canada, July 1995.

1. Let A, B, C, D be distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. O is an arbitrary point on the line XY but not on AD. CO intersects the circle with diameter AC again at M, and BO intersects the other circle again at N. Prove that AM, DN and XY are concurrent.

Soln. The result is trivial if O coincides with X or Y. Suppose it does not. From

$$ON \cdot ON = OX \cdot OY = OC \cdot OM,$$

$BCMN$ is cyclic. If O is on the segment XY, then

$$\angle MAD + \angle MNB + \angle BND = \angle MAD + \angle MCA + \angle AMC = 180^\circ >$$

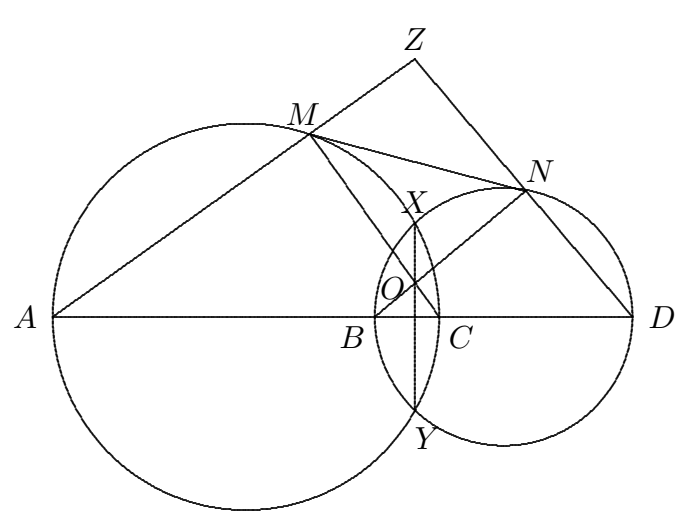
If O is not on the segment XY, then

$$\angle MAD = 180^\circ - \angle AMC - \angle MAC = 180^\circ - \angle BND - \angle ONM = \angle MND.$$

In either case, $ADNM$ is also cyclic. Let AM and DN intersect at Z. Let the line ZX intersect the first circle at Y_1 and the second at Y_2. Then

$$ZX \cdot ZY_1 = ZA \cdot ZM = ZD \cdot ZN = ZX \cdot ZY_2.$$

Hence $Y_1 = Y_2 = Y$ and indeed Z lies on XY.



2. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+b)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Soln. Let S be the right hand side and $T = a(b+c)+b(a+c)+c(a+b) = 2(ab+ac+bc) \geq 6(abc)^{2/3} = 6$. Then by Cauchy's inequality, we have

$$ST \geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 = \frac{T^2}{4}.$$

Thus $S \geq T/4 \geq 3/2$ as desired.

Second soln. Let $x = 1/a$. $y = 1/b$, $z = 1/c$. Then $xyz = a$ and

$$S = \frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}.$$

By Cauchy's inequality,

$$[(x+y) + (z+x) + (x+y)]S \geq (x+y+z)^2$$

or $S \geq \frac{(x+y+z)}{2} \geq \frac{3}{2}(xyz)^{1/3} = \frac{3}{2}$ as desired.

3. Determine all integers $n > 3$ such that there are n points $A_1, A_2, \ldots, A_n$ in the plane which satisfy the following two conditions simultaneously.

(a) No three lie on the same line.

(b) There exist real numbers $p_1, p_2, \ldots, p_n$ such that the area of $\triangle A_iA_jA_k$ is equal to $p_i + p_j + p_k$ for $1 \leq i < j < k \leq n$.

Soln. We claim $n = 4$ is the only answer. For $n = 4$, let $A_1A_2A_3A_4$ be a unit square and let $p_1 = p_2 = p_3 = p_4 = 1/6$. It remains to show no solution exists for $n = 5$ which implies that there no solution for any $n \geq 5$.

Suppose to the contrary that there is solution with $n = 5$. Denote the area of $\triangle A_iA_jA_k$ by $[ijk] = p_i + p_j + p_k$. We cannot have $p_i = p_j$. if for example $p_4 = p_5$, then $[124] = [125]$ and $[234] = [235]$, which implies that A_4A_5 is parallel to both A_1A_2 and A_2A_3. This is impossible since A_1, A_2, A_3 are not collinear. We also observe that if $A_iA_jA_kAA_\ell$ is convex, then $p_i + p_k = p_j + p_\ell$. This follows from $[ijk] + [k\ell i] = [jk\ell] + [\ell ij]$.

Consider the convex hull of $A_1, \ldots, A_5$, we have 3 cases. First, suppose that the convex hull is a pentagon $A_1A_2A_3A_4A_5$. Since $A_1A_2A_3A_4$ and $A_1A_2A_3A_5$ are convex, our observation yields $p_1 + p_3 = p_2 + p_4$ and $p_1 + p_3 = p_2 + p_5$. Hence $p_4 = p_5$, a contradiction.

Next we suppose that the convex hull is a quadrilateral $A_1A_2A_3A_4$. We may assume that A_5 is within $A_3A_4A_1$. Then $A_1A_2A_3A_5$ is convex and we have the same contradiction as before.

Finally suppose that the convex hull is a triangle $A_1A_2A_3$. Since $[124] + [234] + [314] = [125] + [235] + [315]$ we have $p_4 = p_5$, a contradiction.

4. The positive real numbers $x_0, x_1, \ldots, x_{1995}$ satisfy $x_0 = x_{1995}$ and

$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$$

for $i = 1, 2, \ldots, 1995$. Find the maximum value that x_0 can have.

Soln. The given condition is equivalent to

$$(2x_i^2 - x_{i-1})(x_i - \frac{1}{x_{i-1}}) = 0$$

which yield $x_i = x_{i-1}/2$ and $x_i = 1/x_{i-1}$. We shall prove by induction that $x_i = 2^{k_i} x_0^{\epsilon_i}$, where $|k_i| \le i$ and $\epsilon_i = (-1)^{k_i+i}$. This is true for $i = 0$, with $k_0 = 0$, $\epsilon_0 = 1$. Assume that it is true for $i-1$. Then for $x_i = \frac{1}{2}x_{i-1}$, we have $k_i = k_{i-1} - 1$ and $\epsilon_i = \epsilon_{i-1}$. For $x_i = 1/x_{i-1}$, we have $k_i = -k_{i-1}$ and $\epsilon_i = -\epsilon_{i-1}$. In each case, it is immediate that $|k_i| \le i$ and $\epsilon_i = (-1)^{k_i+i}$. Thus $x_0 = x_{1995} = 2^k x_0^\epsilon$ where $k = k_{1995}$ and $\epsilon = \epsilon_{1995}$. Thus $|k| \le 1995$ and $\epsilon = (-1)^{1995+k}$. If k is odd then $\epsilon = 1$ and this is impossible. Thus k is even and $\epsilon = -1$ and $x_0^2 = 2^k$. Since $|k| \le 1995$, we have $k \le 1994$. Hence $x_0 \le 2^{997}$. The value $x_0 = 2^{997}$ can be attained by choosing $x_i = \frac{1}{2}x_{i-1}$ for $i = 1, 2, \ldots, 1994$ and $x_{1995} = 1/x_{1994}$.

5. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$ and $\angle BCD = \angle EFA = \pi/3$. Let G and H be two points interior to the hexagon such that angles AGB and DHE are both $2\pi/3$. Prove that $AG + GB + GH + DH + HE \ge CF$.

Soln. Note that BCD and EFA are equilateral triangles. Hence BE is an axis of symmetry of $ABDE$. Reflect BCD and EFA about BE to $BC'A$ and $EF'D$ respectively. Since $\angle BGA = 180° - \angle AC'B$, G lies on the circumcircle of ABC'. By Ptolemy's Theorem, $AG + GB = C'G$. Similarly, $DH + HE = HF'$. It follows that

$$CF = C'F' \le C'G + GH + HF' = AG + GB + GH + DH + HE,$$

with equality if and only if G and H both lie on $C'F'$.

6. Let p be an odd prime. Find the number of p-element subsets A of $\{1, 2, \ldots, 2p\}$ such that the sum if all elements of A is divisible by p.

Soln. For any p-element subset A of $\{1, 2, \ldots, 2p\}$, denote by $s(A)$ the sum of the elements of A. Of the $\binom{2p}{p}$ such subsets, $B = \{1, 2, \ldots, p\}$ and $C = \{p+1, p+2, \ldots, 2p\}$ satisfy $s(B) = s(C) \equiv 0 \pmod p$. For $A \ne B, C$, we have $A \cap B \ne \emptyset \ne A \cap C$. Partition the $\binom{2p}{p} - 2$ p-element subsets other than B and C into groups of size p as follows. Two subsets A and A' are in the same group if and only if $A' \cap C = A \cap C$ and $A' \cap B$ is a cyclic

permutation of $A \cap B$ within B. Suppose $A \cap B$ has n elements, $0 < n < p$. For some m such that $0 < m < p$,

$$A' \cap B = \{x + m : x \in A \cap B, x + m \leq p\} \cup \{x + m - p : x \in A \cap B, x \leq p < x + m\}.$$

Hence $s(A') - s(A) \equiv mn \pmod{p}$, but mn is not divisible by p. It follows that exactly one subset A in each group satisfies $s(A) \equiv 0 \pmod{p}$, and the total number of such subsets is $p^{-1}\left(\binom{2p}{p} - 2\right) + 2$.

Second soln. Let ω be a primitive pth root of unity. Then

$$a_0 + a_1\omega + \cdots + a_{p-1}\omega^{p-1} = 0 \quad \text{iff} \quad a_0 = a_1 = \cdots = a_{p-1}.$$

Also

$$\prod_{k=1}^{2p}(x - \omega^k) = (x^p - 1)^2 = x^{2p} - 2x^p + 1.$$

If $t(\omega) = \sum a_i\omega^i$, then a_i is the number of combinations with $i_1 + \cdots + i_p \equiv i \pmod{p}$. Therefore

$$(a_0 - 2) + a_1\omega + \cdots + a_{p-1}^{p-1} \equiv 0$$

and $a_0 - 2 = a_1 = \cdots = a_p$. So $|S| = \frac{\binom{2p}{p}-2}{p} + 2$.