Competition corner

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In this issue we publish the problems of 43rd Czech (and Slovak) Mathematical Olympiad, 1994, Ukrainian Mathematical Olympiad, 2002 and 43rd International Mathematical Olympiad held in Glasgow, United Kingdom, July 2002.

Please send your solutions of these olympiads to me at the address given. All correct solutions will be acknowledged. We also present solutions of Canadian Mathematical Olympiad 1993, as well the problems used to select the Singapore Team to the 2002 International Mathematical Olympiad.

Problems

43rd Czech (and Slovak) Mathematical Olympiad, 1994

1. Let $\mathbb{N}$ be the set of all natural numbers and $f : \mathbb{N} \to \mathbb{N}$ a function which satisfies the inequality

$$f(x) + f(x+2) \leq 2f(x+1) \quad \text{for any } x \in \mathbb{N}.$$

Prove that there exists a line in the plane which contains infinitely many points with coordinates $(n, f(n))$.

2. A cube of volume V contains a convex polyhedron M. The perpendicular projection of M into each face of the cube coincides with all of this face. What is the smallest possible volume of the polyhedron M?

3. A convex 1994-gon M is drawn in the plane together with 997 of its diagonals drawn. Each diagonal divides M into two sides. The number of edges on the shorter side is defined to be the length of the diagonal. Is it possible to have

(a) 991 diagonals of length 3 and 6 of length 2?

(b) 985 diagonals of length 6, 4 of length 8 and 8 of length 3?

4. Let $a_1, a_2, \dots$ be an arbitrary sequence of natural numbers such that for each n, the number $(a_n - 1)(a_n - 2)\dots(a_n - n^2)$ is a positive integral multiple of n^{n^2-1}. Prove that for any finite set P of prime numbers, the following inequality holds:

$$\sum_{p \in P} \frac{1}{\log_p a_p} < 1.$$

5. Let AA_1, BB_1, CC_1 be the heights of an acute-angles triangle ABC (i.e., A_1 lies on the line BC and $AA_1 \perp BC$, etc.) and V their intersection. If the triangles AC_1V, BA_1V, CB_1V have the same areas, does it follow that the triangle ABC is equilateral?

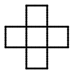
6. Show that from any quadruple of mutually different numbers lying in the interval $(0, 1)$ it's possible to choose two numbers $a \neq b$ in such a way that

$$\sqrt{(1-a^2)(1-b^2)} > \frac{a}{2b} + \frac{b}{2a} - ab - \frac{1}{8ab}.$$

Ukrainian Mathematical Olympiad, 2002

Selected problems.

1. (9th grade) The set of numbers $1, 2, \dots, 2002$ is divided into 2 groups, one comprising numbers with odd sums of digits and the other comprising numbers with even sums odd digits. Let A be the sum of the numbers in the first group and B be the sum of the numbers in the second group. Find $A - B$.

2. (9th grade) What is the minimum number of the figure  that we may mark on the cells of the (8×8) chessboard so that it's impossible to mark more such figures without overlapping?

3. (10th grade) Let A_1, B_1, C_1 be the midpoints of arcs BC, CA, AB of the circumcircle of $\triangle ABC$, respectively. Let A_2, B_2, C_2 be the tangency points of the incircle of $\triangle ABC$, with sides BC, CA, AB, respectively Prove that the lines A_1A_2, B_1B_2, C_1C_2 are concurrent.

4. (10th grade) Find the largest K such that the inequality

$$\frac{1}{(x+y+z)^2} + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{K}{\sqrt{(x+y+z)xyz}}$$

holds for all positive x, y, z.

5. (11th grade) Solve in integers the following equation

$$n^{2002} = m(m+n)(m+2n)\cdots(m+2001n).$$

6. (11th grade) Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x)f(x+y) + 2f(x+2y) + f(2x+y)f(y) = x^4 + y^4 + x^2 + y^2.$$

7. (11th grade) Let C_1, A_1, B_1 be the points at the sides of a given acute $\triangle ABC$ such that $A_1B = A_1C_1$, $A_1C = A_1B_1$. Let I_1 be the incentre of $\triangle A_1B_1C_1$ and H be the orthocentre of $\triangle ABC$. Prove that the points B_1, C_1, I_1, H are concyclic.

8. (11th grade) Let $a_1, a_2, \dots, a_n$, $n \geq 1$, be real numbers ≥ 1 and $A = 1 + a_1 + \cdots + a_n$. Define x_k, $0 \leq k \leq n$ by

$$x_0 = 1, \quad x_k = \frac{1}{1 + a_k x_{k-1}}, \; 1 \leq k \leq n.$$

Prove that

$$x_1 + x_2 + \cdots + x_n > \frac{n^2 A}{n^2 + A^2}.$$