Singapore International Mathematical Olympiad

National Team Selection Test I 2008

Time allowed: 4.5 hours **26 April 2008**

No calculator is allowed

1. In $\triangle ABC$, D is a point on AB and E is a point on AC such that BE and CD are bisectors of $\angle B$ and $\angle C$ respectively. Let Q, M and N be the feet of perpendiculars from the midpoint P of DE onto BC, AB and AC, respectively. Prove that $PQ = PM + PN$.

2. Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 x_2 \cdots x_n = 1$. Prove that

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leq 1.$$

3. Find all odd primes p, if any, so that p divides $\sum_{n=1}^{103} n^{p-1}$.

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4. Let $\mathcal{C}$ be a circle centred at O, and let ABP be a line segment such that A, B lie on $\mathcal{C}$ and P is a point outside $\mathcal{C}$. Let C be a point on $\mathcal{C}$ such that PC is tangent to $\mathcal{C}$ and let D be the point on $\mathcal{C}$ such that CD is a diameter of $\mathcal{C}$ and intersects AB inside $\mathcal{C}$. Suppose that the lines DB and OP intersect at E. Prove that AC is perpendicular to CE.

5. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ so that

$$(x + y)(f(x) - f(y)) = (x - y)f(x + y)$$

for all $x, y \in \mathbb{R}$.

6. Fifty teams participate in a round robin competition over 50 days. Moreover, all the teams (at least two) that show up in any day must play against each other. Prove that on every pair of consecutive days, there is a team that has to play on those two days.

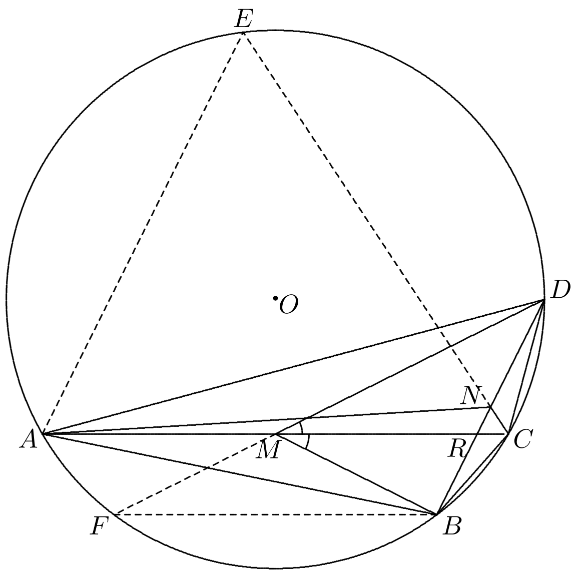
Solution to National Team Selection Test 2007

1. Find all pairs of nonnegative integers (x, y) satisfying $(14y)^x + y^{x+y} = 2007$.

   Solution. Suppose x and y are nonnegative integers satisfying $(14y)^x + y^{x+y} = 2007$. Note that $y \geq 1$ because $y = 0$ does not satisfy the equation. For $x \geq 3$, we have $(14y)^x + y^{x+y} \geq 14^x \geq 14^3 = 2744 > 2007$. Thus $x = 0, 1$ or 2. If $x = 0$, then the equation becomes $1 + y^y = 2007$. That is $y^y = 2006 = (2)(17)(59)$ which obviously has no solution in y. If $x = 1$, then $14y + y^{y+1} = 2007$. That is $y(14 + y^y) = 2007 = 3^2(223)$. Thus $y = 1, 3, 9$ or 223. But none of these satisfy $14y + y^{y+1} = 2007$. If $x = 2$, then $y^2(14^2 + y^y) = 3^2(223)$. Since $y = 1$ does not satisfy this equation, we must have $y = 3$. Consequently $x = 2$ and $y = 3$ and they do satisfy the given equation. Therefore $(x, y) = (2, 3)$ is the only solution to the equation.

2. Let $ABCD$ be a convex quadrilateral inscribed in a circle with M and N the midpoints of the diagonals AC and BD respectively. Suppose AC bisects $\angle BMD$. Prove that BD bisects $\angle ANC$.

   Solution. Let O be the center of the circle. Let the extensions of CN and DM meet the circle at E and F respectively. Join AE and BF.



First note that F is the reflection of B about the line OM so that $MF = MB$. Thus $\angle BFM = \angle BMC$ and FB is parallel to MC. Therefore $\angle BAD = \angle BFM = \angle BMC$. As $\angle ADB = \angle MCB$, we have $\triangle ADB$ is similar to $\triangle MCB$. Therefore, $\frac{BD}{BC} = \frac{DA}{CM} = \frac{DA}{AC/2}$ so that $\frac{AC}{BC} = \frac{DA}{BD/2} = \frac{DA}{NB}$. As $\angle DAC = \angle NBC$, we see that $\triangle DAC$ is similar to $\triangle NBC$ so that $\angle BNC = \angle ADC = \angle AEC$. This means AE is parallel to BD. As NO is perpendicular to BD, we have the extension of NO is perpendicular to AE and meets AE at its midpoint. Thus $\triangle ANE$ is isosceles with $\angle NEA = \angle NAE$. Consequently, $\angle CNR = \angle NEA = \angle NAE = \angle ANR$.

2nd Solution. Set up a coordinate system in which M is the origin, AC is the x-axis and the center of the circle lies on the y-axis with coordinate $(0, d)$. Let the radius of the circle be r and let the gradients of the lines MD and MB be α and $-\alpha$ respectively. Then the coordinates of A and C are $(-\sqrt{r^2 - d^2}, 0)$ and $(\sqrt{r^2 - d^2}, 0)$ respectively. By solving the equation of the circle $x^2 + (y - d)^2 = r^2$ and the equation $y = \alpha x$ of the line MD, we get the coordinate of D equal to

$$\left(\frac{\alpha d+\sqrt{r^2-d^2+\alpha^2r^2}}{1+\alpha^2}, \alpha\left(\frac{\alpha d+\sqrt{r^2-d^2+\alpha^2r^2}}{1+\alpha^2}\right)\right).$$

Replacing α by $-\alpha$, we obtain the coordinate of B as

$$\left(\frac{-\alpha d+\sqrt{r^2-d^2+\alpha^2r^2}}{1+\alpha^2}, -\alpha\left(\frac{-\alpha d+\sqrt{r^2-d^2+\alpha^2r^2}}{1+\alpha^2}\right)\right).$$

Therefore, the midpoint N of BD has coordinate $\left(\frac{\sqrt{r^2-d^2+\alpha^2r^2}}{1+\alpha^2}, \frac{\alpha^2 d}{1+\alpha^2}\right)$. Let AC intersect BD at R. Then the x-coordinate of R is found to be $\frac{r^2-d^2}{\sqrt{r^2-d^2+\alpha^2r^2}}$. Thus a direct calculation gives

$$\left(\frac{RA}{RC}\right)^2=\frac{2r^2-2d^2+\alpha^2r^2+2\sqrt{(r^2-d^2)(r^2-d^2+\alpha^2r^2)}}{2r^2-2d^2+\alpha^2r^2-2\sqrt{(r^2-d^2)(r^2-d^2+\alpha^2r^2)}}=\left(\frac{NA}{NC}\right)^2.$$

That is $\frac{RA}{RC}=\frac{NA}{NC}$. Using the Angle Bisector Theorem, we have $\angle ANB=\angle CNB$.

3. Let $a_1, a_2, \ldots, a_8$ be 8 distinct points on the circumference of a circle such that no three chords, each joining a pair of the points, are concurrent. Every 4 of the 8 points form a quadrilateral which is called a **quad**. If two chords, each joining a pair of the 8 points, intersect, the point of intersection is called a **bullet**. Suppose some of the bullets are coloured red. For each pair (i,j), with $1\le i<j\le 8$, let $r(i,j)$ be the number of quads, each containing a_i, a_j as vertices, whose diagonals intersect at a red bullet. Determine the smallest positive integer n such that it is possible to colour n of the bullets red so that $r(i,j)$ is a constant for all pairs (i,j).

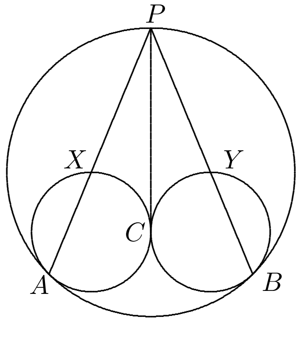
Solution. First note that each bullet is determined by a unique quad. Thus a red quad is a quad whose associated bullet is red. For such an n, let the constant $r(i,j)$ be t. Form an incidence matrix whose rows are indexed by pairs (i,j) and whose columns are indexed by the n red quads. The $((ij),(pqrs))$ entry is 1 if $\{i,j\}\subseteq\{p,q,r,s\}$ and is 0 otherwise. Then each row has t 1's and each column has 6 1's. Thus $6n=t\binom{8}{2}$, i.e., $3n=14t$. The smallest n is $n=14$ and $t=3$.

The following 14 red quads have the desired property:

(1234),(2345),(3456),(4567),(5678),(6781),(7812),
(8123),(1256),(2367),(3478),(4581),(1357),(2468)

4. Two circles Γ_1 and Γ_2 touch externally at the point C and internally at points A and B respectively with another circle Γ centred at O. Suppose the common tangent of Γ_1 and Γ_2 at C meets Γ at P with $PA=PB$. Prove that PO is perpendicular to AB.

Solution. Let r_1, r_2 and r be the radii of Γ_1, Γ_2 and Γ respectively. It suffices to prove that $r_1=r_2$. Then P, O, C are collinear and PO is perpendicular to AB.



First $AX/AP = r_1/r$ so that $PX/PA = (r - r_1)/r$. Also $PC^2 = PX \cdot PA$. Thus $PA^2/PC^2 = PA/PX = r/(r - r_1)$. Therefore, $PA = PC\sqrt{\frac{r}{r-r_1}}$. Similarly, $PB = PC\sqrt{\frac{r}{r-r_1}}$. As $PA = PB$, we have $r_1 = r_2$.

5. Prove the inequality

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \le \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{i<j} a_i a_j$$

for all positive real numbers $a_1, a_2, \ldots, a_n$.

Solution. Let $S = \sum_i a_i$. Denote by L and R the expressions on the left and right hand side of the proposed inequality. We transform L and R using the identity

$$\sum_{i<j} (a_i + a_j) = (n-1) \sum_i a_i. \tag{1}$$

And thus:

$$L = \sum_{i<j} \frac{a_i a_j}{a_i + a_j} = \sum_{i<j} \frac{1}{4}\left(a_i + a_j - \frac{(a_i - a_j)^2}{a_i + a_j}\right) = \frac{n-1}{4} \cdot S - \frac{1}{4} \sum_{i<j} \frac{(a_i - a_j)^2}{a_i + a_j}. \tag{2}$$

To express R we express the sum $\sum\limits_{i<j} a_i a_j$ in two ways; in the second transformation identity (1) will be applied to the squares of the numbers a_i:

$$\sum_{i<j} a_i a_j = \frac{1}{2}\left(S^2 - \sum_i a_i^2\right);$$

$$\sum_{i<j} a_i a_j = \frac{1}{2} \sum_{i<j} \left(a_i^2 + a_j^2 - (a_i - a_j)^2\right) = \frac{n-1}{2} \sum_i a_i^2 - \frac{1}{2} \sum_{i<j} (a_i - a_j)^2.$$

Multiplying the first of these equalities by $n - 1$ and adding the second one we obtain

$$n \sum_{i<j} a_i a_j = \frac{n-1}{2} \cdot S^2 - \frac{1}{2} \sum_{i<j} (a_i - a_j)^2.$$

Hence

$$R = \frac{n}{2S} \sum_{i<j} a_i a_j = \frac{n-1}{4} \cdot S - \frac{1}{4} \sum_{i<j} \frac{(a_i - a_j)^2}{S}. \tag{3}$$

Now compare (2) and (3). Since $S \ge a_i + a_j$ for any $i < j$, the claim $L \ge R$ results.

2nd Solution. Let $S = a_1 + a_2 + \cdots + a_n$. For $i \ne j$,

$$\frac{4a_i a_j}{a_i + a_j} = a_i + a_j - \frac{(a_i - a_j)^2}{a_i + a_j} \le a_i + a_j - \frac{(a_i - a_j)^2}{S} = \frac{\sum\limits_{k \ne i} a_i a_k + \sum\limits_{k \ne j} a_j a_k + 2a_i a_j}{S}.$$

The statement is obtained by summing up these inequalities for all pairs i, j:

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} = \frac{1}{2} \sum_i \sum_{j \ne i} \frac{a_i a_j}{a_i + a_j} \le \frac{1}{8S} \sum_i \sum_{j \ne i} \left(\sum_{k \ne i} a_i a_k + \sum_{k \ne j} a_j a_k + 2a_i a_j\right)$$

$$
\begin{aligned}
&= \frac{1}{8S}\left(\sum_k\sum_{i\neq k}\sum_{j\neq i} a_ia_k + \sum_k\sum_{j\neq k}\sum_{i\neq j} a_ja_k + 2\sum_i\sum_{j\neq i} a_ia_j\right)\\
&= \frac{1}{8S}\left(\sum_k\sum_{i\neq k}(n-1)a_ia_k + \sum_k\sum_{j\neq k}(n-1)a_ja_k + 2\sum_i\sum_{j\neq i} a_ia_j\right)\\
&= \frac{n}{4S}\sum_i\sum_{j\neq i} a_ia_j = \frac{n}{2S}\sum_{i<j} a_ia_j.
\end{aligned}
$$

6. Let A, B, C be 3 points on the plane with integral coordinates. Prove that there exists a point P with integral coordinates distinct from A, B and C such that the interiors of the segments PA, PB and PC do not contain points with integral coordinates.

Solution. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ where $a_1, a_2, b_1, b_2, c_1, c_2$ are integers. If a lattice point $X = (x_1, x_2)$ is between $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then there exist relatively prime positive integers u and v such that $(u + v)x_i = ua_i + vb_i$, $i = 1, 2$. That is $(u + v)(x_i - b_i) = u(a_i - b_i)$, $i = 1, 2$. This implies that $(u + v)$ divides $a_i - b_i$, for $i = 1, 2$. That is $a_1 - b_1$ and $a_2 - b_2$ have a common factor. Therefore, it suffices to find a lattice point $P = (x_1, x_2)$ such that $\gcd(x_1 - a_1, x_2 - a_2) = 1$, $\gcd(x_1 - b_1, x_2 - b_2) = 1$ and $\gcd(x_1 - c_1, x_2 - c_2) = 1$.

Let $p = 2$ or 3. There are at least 4 distinct values (m_1, m_2) modulo p. Thus there exists (r_1, r_2) such that $(r_1, r_2) \not\equiv (a_1, a_2), (b_1, b_2), (c_1, c_2) \pmod 2$. Similarly, there exists (s_1, s_2) such that $(s_1, s_2) \not\equiv (a_1, a_2), (b_1, b_2), (c_1, c_2) \pmod 3$. By the Chinese Remainder Theorem, there exists (p_1, p_2) such that $p_1 \equiv r_1 \pmod 2$, $p_1 \equiv s_1 \pmod 3$; and $p_2 \equiv r_2 \pmod 2$, $p_2 \equiv s_2 \pmod 3$. We can also require $p_1 \neq a_1, b_1, c_1$ and $p_2 \neq a_2, b_2, c_2$. This ensures that $(p_1, p_2) \not\equiv (a_1, a_2)$, (b_1, b_2), $(c_1, c_2) \pmod 2$ and $\pmod 3$.

Suppose p is a prime larger than 3 such that p divides $p_2 - a_2$, $p_2 - b_2$, or $p_2 - c_2$. There are only a finite numbers of such primes p. Let the set of all such primes be Y. For each $p \in Y$, pick an integer $z_p \not\equiv a_1, b_1, c_1 \mod p$. Since $p > 3$, this is possible. Apply the Chinese Remainder Theorem to find an x_1 such that $x_1 \equiv p_1 \pmod 2$, $x_1 \equiv p_1 \pmod 3$ and $x_1 \equiv z_p \pmod p$, $p \in Y$. Take $x_2 = p_2$. Then $P = (x_1, x_2)$ is the desired point.