Singapore International Mathematical Olympiad 2005
Senior Team Training

Take Home Test Solutions

1. Determine which of the numbers

$$\frac{2001^{1111}+1}{2001^{2222}+1} \quad \text{and} \quad \frac{2001^{2222}+1}{2001^{3333}+1}$$

is larger.

Let $a = 2001^{1111}$, $A = \frac{a+1}{a^2+1}$ and $B = \frac{a^2+1}{a^3+1}$. Then,

$$\frac{A}{B} = \frac{(a+1)(a^3+1)}{(a^2+1)^2} = \frac{a^4+(a^3+a)+1}{(a^2+1)^2} > \frac{a^4+2a^2+1}{(a^2+1)^2} = 1.$$

Here, the last inequality uses $AM \geq GM$. Alternatively,

$$A - B = \frac{a^3+a-2a^2}{(a^2+1)(a^3+1)} > 0.$$

2. Determine, with proof, the set of all positive intergers that cannot be represented in the form $\frac{a}{b} + \frac{a+1}{b+1}$ for some positive integers a and b.

Let A denote the given expression. Then $A = \frac{2ab+a+b}{b(b+1)}$ implies $b \mid a$. Let $a = mb$, $m \in \mathbb{N}$. Then $A = m + \frac{mb+1}{b+1} = 2m - \frac{m-1}{b+1}$ implies $b+1 \mid m-1$. Let $m-1 = n(b+1)$, $n \geq 0$, $n \in \mathbb{Z}$. Then $A = n(2b+1)+2$. If $n = 0$, $A = 2$. If $n = 1$, then by varying b, we get $A = 5, 7, \ldots$. We also note that $A \neq 0, 3$. We are left with even numbers $x > 2$. $A = x$ if and only if $x - 2 = n(2b+1)$ if and only if $x - 2$ is a multiple of some odd prime. Thus the required set is $\{1, 2^k + 2 : k \in \mathbb{N}\}$.

3. Let x, y, z be positive numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq \sqrt{3}.$$

Let S be the LHS. We shall find the smallest value of S.

$$\begin{aligned} S^2 &= \frac{x^2y^2}{z^2} + \frac{y^2z^2}{x^2} + \frac{z^2x^2}{y^2} + 2x^2 + 2y^2 + 2z^2 \\ &= \frac{1}{2}\left(\frac{x^2y^2}{z^2} + \frac{z^2x^2}{y^2}\right) + \frac{1}{2}\left(\frac{z^2x^2}{y^2} + \frac{y^2z^2}{x^2}\right) + \frac{1}{2}\left(\frac{y^2z^2}{x^2} + \frac{x^2y^2}{z^2}\right) + 2 \\ &\geq x^2 + y^2 + z^2 + 2 = 3. \end{aligned}$$

When $x = y = x = 1/\sqrt{3}$, equality holds.

4. Let A and B be two sets of N consecutive integers. If $N = 2005$, is it possible to arrange A and B into sequences $A = (a_1, a_2, \ldots, a_N)$, $B = (b_1, b_2, \ldots, b_N)$ in some order (the orders for A and B may be different) so that the sequence of sums $(a_1 + b_1, a_2 + b_2, \ldots, a_N + b_N)$ is a sequence of N consecutive integers? What if $N = 2006$?

Let $A = \{a+1, a+2, \ldots, a+N\}$ and $B = \{b+1, b+2, \ldots, b+N\}$. Suppose the arrangement is possible so that $(a_1 + b_1, a_2 + b_2, \ldots, a_N + b_N) = (m + 1, m + 2, \ldots, m + N)$. Then

$$\begin{aligned} a + 1 + \cdots + a + N + b + 1 + \cdots + b + N &= m + 1 + \cdots + m + N \\ N(a + b) + \frac{N(N + 1)}{2} &= Nm \\ (a + b) + \frac{N + 1}{2} &= m. \end{aligned}$$

Thus, N must be odd. Hence the arrangement is impossible if $N = 2006$. If $N = 2005$, arrange

A as	$a + 1,$	$a + 3,$	$\ldots,$	$a + 2005,$	$a + 2,$	$a + 4,$	$\ldots,$	$a + 2004$
B as	$b + 1003,$	$b + 1002,$	$\ldots,$	$b + 1,$	$b + 2005,$	$b + 2004,$	$\ldots,$	$b + 1004.$

5. Let k be an odd positive integer. If $(2 + \sqrt{3})^k = m + n\sqrt{3}$, where m and n are positive integers, show that $m - 1$ is a perfect square.

Write the odd positive integer k as $2j - 1$ and let $(2 + \sqrt{3})^{2j-1} = m_j + n_j\sqrt{3}$. We first obtain a recurrence relation for m_j. Note that

$$m_{j+1} + n_{j+1}\sqrt{3} = (m_j + n_j\sqrt{3})(2 + \sqrt{3})^2. \tag{1}$$

Hence

$$m_{j+1} = 7m_j + 12n_j \quad \text{and} \quad n_{j+1} = 4m_j + 7n_j.$$

Therefore, $m_{j+2} = 97m_j + 168n_j$. Eliminating n_j using this and equation (1) gives

$$m_{j+2} = 14m_{j+1} - m_j.$$

Of course, $m_1 = 1$, $m_2 = 26$. Define a sequence (c_j) by $c_1 = 1$, $c_2 = 5$ and $c_{j+2} = 4c_{j+1} - c_j$. We wish to show that $m_j = c_j^2 + 1$ for all j. In the course of doing this, we need to verify first

Claim. $c_{j+1}^2 + c_j^2 = 4c_{j+1}c_j + 6$.
This is proved by induction. It holds clearly for $j = 1$. Suppose it holds for some j. Then

$$\begin{aligned} c_{j+2}^2 + c_{j+1}^2 &= 17c_{j+1}^2 + c_j^2 - 8c_{j+1}c_j \\ &= 15c_{j+1}^2 - c_j^2 + 12. \end{aligned}$$

Also,

$$\begin{aligned} 4c_{j+2}c_{j+1} + 6 &= 16c_{j+1}^2 - 4c_{j+1}c_j + 6 \\ &= 15c_{j+1}^2 - c_j^2 + 12. \end{aligned}$$

This completes the proof of the claim by induction.

Now we will show that $m_j = c_j^2 + 1$ by induction on j. For $j = 1, 2$, this is obviously true. Now

$$\begin{aligned}
c_{j+2}^2 + 1 &= 16c_{j+1}^2 - 8c_{j+1}c_j + c_j^2 + 1 \\
&= 14c_{j+1}^2 - c_j^2 + 13 \quad \text{by the claim} \\
&= 14(m_{j+1} - 1) - (m_j - 1) + 13 \quad \text{by the inductive hypothesis} \\
&= 14m_{j+1} - m_j = m_{j+2}.
\end{aligned}$$

6. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that (1) if $x < y$, then $f(x) < f(y)$ and, (2) $f(y(f(x)) = x^2 f(xy)$ for all $x, y \in \mathbb{N}$. Here $\mathbb{N}$ denotes the set of all positive integers.

Note that from (1), $x = y$ if $f(x) = f(y)$ (the function is *injective*). Put $y = 1$ in (2). Then $f(f(x)) = x^2 f(x)$. Put $y = f(z)$ in (2). Then

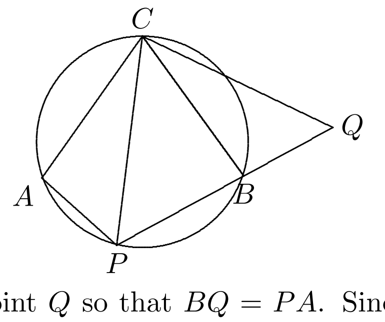
$$f(f(z)f(x)) = x^2 f(xf(z)) = x^2 z^2 f(zx) = f(zx).$$

Thus $f(z)f(x) = f(zx)$. We claim that $f(m) = m^2$ for all m. Suppose not. These is some m so that $f(m) \neq m^2$.

Case 1. $f(m) > m^2$.
Then $m^2 f(m) = f(f(m)) > f(m^2) = (f(m))^2$. Hence $m^2 > f(m)$, a contradiction.

Case 2. $f(m) < m^2$.
Then $m^2 f(m) = f(f(m)) < f(m^2) = (f(m))^2$. Hence $m^2 < f(m)$, a contradiction.

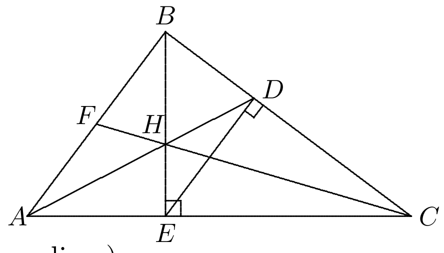
7. The triangle ABC has $CA = CB$. P is a point on the circumcircle of triangle ABC lying on the arc between A and B not containing C. D is the foot of the perpendicular from C to PB. Show that $PA + PB = 2 \cdot PD$.



Extend PB beyond B to a point Q so that $BQ = PA$. Since $BQ = PA$, $CB = CA$ and $\angle CAP = \angle CBA$, the triangles CAP and CBQ are congruent. Thus $\angle CPA = \angle CQB$. The angles $\angle CPA$ and $\angle CPB$ are subtended by chords of equal length and hence are equal. Therefore, $\angle CPB = \angle CQB$. It follows that the triangle CPQ is isoceles with $CP = CQ$. Hence $2 \cdot PD = PQ = PB + BQ = PB + PA$.

8. In triangle ABC, E is the foot of the perpendicular from B onto AC, D is the foot of the perpendicular from E onto BC, and F is the midpoint of AB. Suppose AD, BE and CF intersect at a common point H. Prove that $\angle ABC = 90°$.

Solution By Menelaus' theorem, we have $AE/EC = BD/CD$. As $\triangle CDE$ is similar to $\triangle EDB$, we have $AE = BD(EC/CD) = BD(BE/ED)$. Hence, $AE/BE = BD/ED$. Also, $\angle AEB = \angle BDE$. Thus, $\triangle AEB$ is similar to $\triangle BDE$. Therefore, $\angle ABC = \angle ABE + \angle EBD = \angle BED + \angle EBD = 90^\circ$.

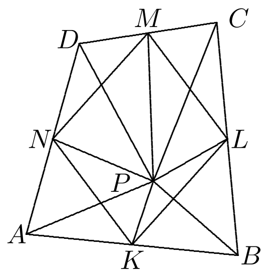


(Note that H must lie inside $\triangle ABC$ as CF is a median.)
The converse of the problem is also true. That is: if $\angle ABC = 90^\circ$, then AD, BE and CF are concurrent. This can be seen as follows: Let CF intersect BE and DE at H and G respectively. Join AH and DH. Now, $\angle ABC = 90^\circ$ implies that AB is parallel to ED so that $\triangle BFH$ is similar to EGH. Hence $BF/BH = EG/EH$. Therefore, $BA/BH = 2BF/BH = 2EG/EH = ED/EH$. That is $\triangle ABH$ is similar to $\triangle EDH$. Therefore, A, H, D are collinear. Thus, AD, BE and CF are concurrent.

9. P is a point on the plane inside a convex quadrilateral $ABCD$. The bisectors of $\angle APB$, $\angle BPC$, $\angle CPD$ and $\angle DPA$ meet the lines AB, BC, CD and DA at K, L, M and N respectively. If $KLMN$ is a parallelogram. Prove that $PB = PD$ and $PA = PC$.

Solution Suppose $KLMN$ is a parallelogram. The angle bisector theorem implies that

$$\frac{AK}{KB}\frac{BL}{LC}\frac{CM}{MD}\frac{DN}{NA} = \frac{PA}{PB}\frac{PB}{PC}\frac{PC}{PD}\frac{PD}{DA} = 1.$$



Suppose KN meets BD at Q. Apply Menelaus' theorem to to $\triangle ABD$ and the line KNQ, we have

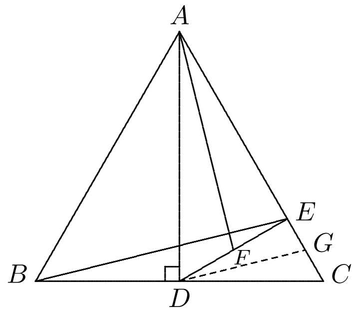
$$\frac{DN}{NA}\frac{AK}{KB}\frac{BQ}{QD} = 1.$$

Substituting this into the first equation, we have

$$\frac{BL}{LC}\frac{CM}{MD}\frac{DQ}{QB} = 1.$$

By Menelaus' theorem applied to $\triangle BCD$ and the points Q, M, L, we see that Q, M, L are collinear. That is KN meets LM at Q, contradicting that fact that LM is parallel to KN. Hence, KN, BD and LM are all parallel. Therefore, $\frac{PA}{PB} = \frac{AK}{KB} = \frac{AN}{ND} = \frac{PA}{PD}$. That is $PB = PD$. Similarly, $PA = PC$.

10. In the acute triangle ABC, let D be the foot of the perpendicular from A to BC, let E be the foot of the perpendicular from D to AC, and let F be a point on the line segment DE. Prove that AF is perpendicular to BE if and only if $FE/FD = BD/DC$.



Solution Let G be the point on CE such that DG is parallel to BE. Then $\angle EBD = \angle GDC$. Also $EG/GC = BD/DC$. Note that $\triangle ADE$ is similar to $\triangle DCE$.

Then,

$$\begin{aligned} & FE/FD = BD/DC \\ \Longleftrightarrow \quad & EG/GC = FE/FD \\ \Longleftrightarrow \quad & \triangle ADF \text{ is similar to } \triangle DCG \\ \Longleftrightarrow \quad & \angle DAF = \angle GDC \\ \Longleftrightarrow \quad & \angle DAF = \angle EBD \\ \Longleftrightarrow \quad & AF \perp BE. \end{aligned}$$

Take Home Test – Part 2

1. If x and y are integers, find the smallest positive integer a for which

$$5213x + 8421y = 2005002 + a$$

is possible.

Solution:

By Euclidean algorithm, we find that

$$\gcd(5213, 8421) = 401.$$

Hence

$$\begin{aligned} & 2005002 + a \equiv 0 \pmod{401} \\ \Rightarrow \quad & 2 + a \equiv 0 \pmod{401} \\ \Rightarrow \quad & a \equiv -2 \pmod{401}. \end{aligned}$$

So the smallest positive integer value for a is $401 - 2 = 399$.

2. Find all positive integers p and q such that

$$\frac{1}{p} - \frac{1}{q} = \frac{3}{40}.$$

Solution:

$$\begin{aligned} & \frac{1}{p} - \frac{1}{q} = \frac{3}{40} \\ \Rightarrow\ & \frac{q-p}{pq} = \frac{3}{40} \\ \Rightarrow\ & 3pq + 40p - 40q = 0 \\ \Rightarrow\ & p(3q+40) - \frac{40}{3}(3q+40) + \frac{40^2}{3} = 0 \\ \Rightarrow\ & 3p(3q+40) - 10(3q+40) = -40^2 \\ \Rightarrow\ & (3p-40)(3q+40) = -40^2 \\ \Rightarrow\ & (40-3p)(3q+40) = 40^2 \end{aligned}$$

Since $40^2 = (2^3 \times 5)^2 = 2^6 \times 5^2$ and $3q + 40 > 40$, $(40 - 3p, 3q + 40)$ must be one of the followings:

$$\begin{aligned} &(1,1600),(2,800),(4,400),(5,320),(8,200), \\ &(10,160),(16,100),(20,80),(25,64),(32,50). \end{aligned}$$

Solving for integer values for (p, q), we have

$$(p,q) = (13,520),(12,120),(10,40),(8,20),(5,8).$$

3. Show that the equation $15x^2 - 7y^2 = 9$ has no solution in integers.

   Solution:

   If the equation has a solution in integer, then

$$\begin{aligned} & 15x^2 - 7y^2 = 9 \\ \Rightarrow\ & -y^2 \equiv 0 \pmod 3 \\ \Rightarrow\ & y \equiv 0 \pmod 3. \end{aligned}$$

   Hence $y = 3y_1$ for some integer y_1. This implies that

$$\begin{aligned} & 15x^2 - 7(3y_1)^2 = 9 \\ \Rightarrow\ & 5x^2 - 21y_1^2 = 3 \\ \Rightarrow\ & 2x^2 \equiv 0 \pmod 3 \\ \Rightarrow\ & x \equiv 0 \pmod 3. \end{aligned}$$

   Hence $x = 3x_1$ for some integer x_1. This implies that

$$\begin{aligned} & 15(3x_1)^2 - 7(3y_1)^2 = 9 \\ \Rightarrow\ & 15x_1^2 - 7y_1^2 = 1 \\ \Rightarrow\ & -y_1^2 \equiv 1 \pmod 3 \\ \Rightarrow\ & y_1^2 \equiv 2 \pmod 3. \end{aligned}$$

   The last congruence is impossible.

   Hence the given equation has no solution in integers.

4. Show that if $m < n$, then $2^{2^m} + 1$ divides $2^{2^n} - 1$. Hence deduce that $2^{2^m} + 1$ and $2^{2^n} + 1$ are relatively prime. Conclude that there are infinitely many primes.

Solution:

If $m < n$ then $n = m + k$ for some integer $k \geq 1$, so we have

$$2^{2^n} - 1 = 2^{2^m 2^k} - 1 = (2^{2^m} + 1)(2^{2^m(2^k-1)} - 2^{2^m(2^k-2)} + - \cdots + 2^{2^m} - 1).$$

Hence $2^{2^m} + 1$ divides $2^{2^n} - 1$.

Let $d = \gcd(2^{2^m} + 1, 2^{2^n} + 1)$. By above, we have

$$\begin{aligned} & 2^{2^n} + 1 = (2^{2^n} - 1) + 2 = l(2^{2^m} + 1) + 2 \text{ for some integer } l \\ \Rightarrow \quad & d | 2 \quad (\text{since } d | (2^{2^m} + 1) \text{ and } d | (2^{2^n} + 1)) \\ \Rightarrow \quad & d = 1 \text{ or } d = 2 \\ \Rightarrow \quad & d = 1 \quad (\text{since } 2^{2^m} + 1 \text{ is odd}) \end{aligned}$$

Thus $\gcd(2^{2^m} + 1, 2^{2^n} + 1) = 1$, i.e. $2^{2^m} + 1$ and $2^{2^n} + 1$ are relatively prime.

For any positive integer n, let p_n be a prime divisor of $2^{2^n} + 1$. For any m, n with $n \neq m$, we have $p_n \neq p_m$ since $2^{2^n} + 1$ and $2^{2^m} + 1$ are relatively prime. Hence $\{p_1, p_2, p_3 \ldots\}$ is an infinite set of primes.