# TEACHING EULER GROUPS BY DISCOVERY METHOD

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## What is the Discovery Method?

Some Quotations:

1) What is good teaching?

   Giving opportunity to the student to discover things by himself.

   Herbert Spencer

2) The ideas should be born in the student's mind and the teacher should act only as midwife.

   Socrates

3) Nothing is more important than to see the sources of invention which are, in my opinion, more important than the inventions themselves.

   Leibniz

4) First guess, then prove -- that's the way to do it. When introduced at the wrong time or place, good logic may be the worst enemy of good teaching. Beauty in mathematics is seeing the truth without effort.

   George Polya

5) To teach new concepts without motivation is to torture inquisitive minds and to murder would be mathematicians

   One of our colleagues

## Teaching Group Theory in School

Since 1950, there has been a world wide reform on mathematical education. Many new notions and results of mathematics are being promoted for inclusion in the school syllabuses. One of them is the concept of a group.

The purpose of this article is to suggest one way of teaching students about a certain class of groups (Euler groups) by the so-called Discovery Method. Euler groups are finite commutative groups, whose motivation comes from the study of natural numbers. The concept of an Euler group is logically simple enough to be understood by the high school students, and yet its structure is interesting enough to stimulate the curiosity of the students.

Teaching Euler Groups by Discovery Method

Teacher: Let us play a Number-Game; take out a piece of paper.

Student: Done.

Teacher: Write down a positive integer n greater than 1.

Student writes: 15

Teacher: Write down the set of all positive integers less than n.

Student writes: $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$ .

Teacher: Strike out those integers in your set which have common divisors with n, leaving only those integers which are coprime to n.

Student writes: C$\{1,2,4,7,8,11,13,14\}$ .

Teacher: Construct a new multiplication table with your set of integers using the following rule: For each a, b we define

$$a * b = \text{the remainder of } (a \times b) \div n$$

where, in your case, n = 15

Student writes:

| * | 1 | 2 | 4 | 7 | 8 | 11 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 7 | 8 | 11 | 13 | 14 |
| 2 | 2 | 4 | 8 | 14 | 1 | 7 | 11 | 13 |
| 4 | 4 | 8 | 1 | 13 | 2 | 14 | 7 | 11 |
| 7 | 7 | 14 | 13 | 4 | 11 | 2 | 1 | 8 |
| 8 | 8 | 1 | 2 | 11 | 4 | 13 | 14 | 7 |
| 11 | 11 | 7 | 14 | 2 | 13 | 1 | 8 | 4 |
| 13 | 13 | 11 | 7 | 1 | 14 | 8 | 4 | 2 |
| 14 | 14 | 13 | 11 | 8 | 7 | 4 | 2 | 1 |

Teacher: Observe carefully. Do you find any integer in your table which is not in your set?

Student: No. For each a, b in my set, a * b is again in my set, I do not need any new integer to construct my table.

Teacher: In mathematics, when we come across a set having an operation defined "within" its own elements, the set is called a "closed" mathematical system. Now, let us try to study the "mathematical structure" of your system. Observe your table carefully and tell me any interesting properties you may find.

Student: The table is diagonally symmetric, in the sense that for each a, b in my set, a * b and b * a are equal.

Teacher: That is right. In mathematics we say the operation is "commutative", i.e. for each $a, b \in \Phi(n)$,

$$a * b = b * a .$$

Student: I also notice that 1 has the property that for each a in my set

$$1 * a = a.$$

Teacher: That is right. In mathematics we say that 1 acts as the identity in the system.

Student: I also notice that in every row the integers in my set appear once and only once. In particular, 1 appears once and only once in every row. This means that for each integer a in my set there exists one and only one b in my set with a * b = 1.

Teacher: That is right. In mathematics we say that every element in your system has an "inverse" and if $a * b = 1$ we denote b by $a^{-1}$ or denote a by $b^{-1}$.

Student: Have I discovered all the properties you want me to?

Teacher: Not yet. There is one more important property I would like you to discover. However, this property is "hidden" inside the table, and not easily perceived.

Student: May I make some guesses?

Teacher: You may try.

Student: Am I right to say that in my system I can do "division" in the sense that for each given a and b, I can always find a unique element c such that

$$a * c = b, \text{ which}$$

means that $b \div a = c$ ?

Teacher: The fact that the elements in your set appear exactly once in every row does imply that you can do "division" in your set system. However, this is not the property I have in mind.

Student: Can you give me a hint?

Teacher: The property I want you to discover is similar to a property we often come across when we add integers or multiply integers. This property fails when we do division.

Student: Is it the associativity of the operation * ? Am I right to say that for each a, b, c in my set,

$$(a * b) * c = a * (b * c)?$$

Teacher: You are right. Can you "see" this property in your table?

Student: I can't, the table does not seem to have any pattern or rule to suggest that the operation * is associative.

Teacher: In this case, what will you do to convince that the operation is indeed associative?

Student: The only method I can think of is by trial and error.

Teacher: All right, go and try.

Student writes:

1. $\left.\begin{array}{l}(2 * 4) * 7 = 8 * 7 = 11 \\ 2 * (4 * 7) = 2 * 13 = 11\end{array}\right\} \therefore (2 * 4) * 7 = 2 * (4 * 7)$

$$(4 * 4) * 14 = 1 * 14 = 14, \quad 4 * (4 * 14) = 4 * 11 = 14 \quad \therefore (4 * 4) * 14 = 4 * (4 * 14)$$

$$(11 * 8) * 14 = 13 * 14 = 2, \quad 11 * (8 * 14) = 11 * 7 = 2 \quad \therefore (11 * 8) * 4 = 11 * (8 * 14)$$

Teacher: Are you now convinced that the operation * is associative?

Student: I am almost convinced, when I have more time I will try to convince myself, and to find a proof of this property.

Teacher: That is the right attitude in studying mathematics. Although trial and error can strengthen your belief in something, it always leave room for doubt. Only a good mathematical proof will once and for all put you on a firm foundation for what you believe.

Student: Let me summarize the properties we have discovered:

1. For each $a, b \in \bar{\phi}(n)$, $a * b \in \bar{\phi}(n)$,

   i.e. $< \bar{\phi}(n), * >$ forms a closed system.

2. The operation * is commutative in $< \phi(n), * >$.

3. The operation * is associative in $< \phi(n), * >$.

4. The system $< \phi(n), * >$ has an identity, namely 1.

5. Every element of $\phi(n)$ has a unique inverse.

Teacher: In mathematics we call such a system a commutative group. In this particular case, it is called an Euler group, after the famous mathematician Leonard Euler (1707 - 1783).

Student: Am I right to say that for every given positive integer n, we can construct an Euler group $< \bar{\phi}(n), * >$? And since there are infinitely many positive integers, there are infinitely many Euler groups?

Teacher: You are right. All Euler groups have the five properties you just summarized.

Student: Do Euler groups differ from each other in any way?

Teacher: Yes, they do. One way to compare their structures is to draw their structural graphs. Let me illustrate the method on Euler group 15.

<u>Step 1</u> Write down the set

(15) = {1,2,4,7,8,11,13,14} .

<u>Step 2</u> The first element is 1.

Draw a small circle and put the identity 1 in it. Then delete 1 from the set {~~1~~,2,4,7,8,11,13,14} .

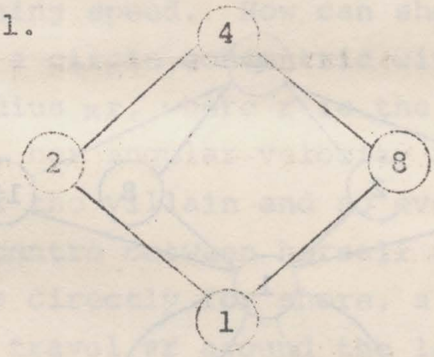
<u>Step 3</u> Now, the first integer is 2. We now compute the products

2, 2 * 2, (2 * 2) * 2, ...,

until the sequence comes back to 1:

2, 2 * 2 = 4, 4 * 2 = 8, 8 * 2 = 1.

We then draw a circuit containing the integers 2,4,8,1.




Then delete 2,4,8, from the above set

e.g. {~~1~~,~~2~~,~~4~~,7,~~8~~,11,13,14} .

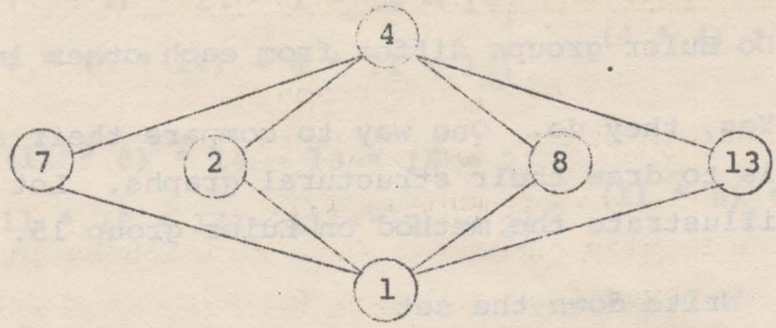
<u>Step 4</u> Now, the first integer is 7. We next compute the sequence

7, 7 * 2, (7 * 7) * 7, ...,

until it comes back to 1, getting

7, 7 * 7 = 14, 4 * 7 = 13, 13 * 7 = 1.

We then draw a circuit containing these integers 7, 4, 13, 1 as follows:



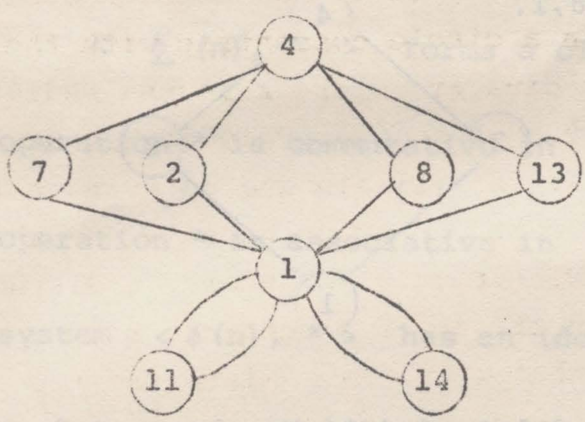

As before, we delete 7 and 13 from the above set, giving

{~~1~~,~~2~~,~~4~~,~~7~~,~~8~~,11,~~13~~,14}

<u>Step 5</u> We now consider the first integer in the new set, which is 11, and find the sequence

11, 11 * 11, (11 * 11) * 11, ...,

Finally, we obtain the following graph representing the structure of the Euler group $< \Phi\ (15), * >$




Student: Euler group has a beautiful structural graph. Are all structural graphs of Euler groups so beautiful?

Teacher: The best way to get the answer is for you to go home and construct as many structural graphs of Euler groups as you can.