THE HOMICIDAL CHAUFFEUR AND OTHER GAMES

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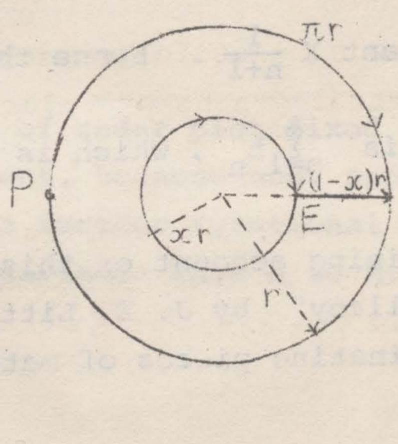
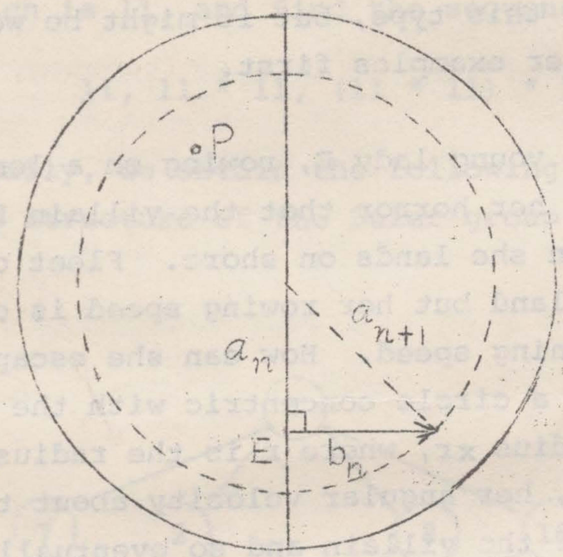
Most people play games for fun, but mathematically a game exists whenever there are two (or perhaps more) players with conflicting objectives, which each tries to achieve while abiding by the rules of the game. The mathematical interest is to see whether one player has a winning strategy, and if so to find it. For obvious reasons, games of pursuit and evasion have been extensively studied; in these, one player P, the pursuer, seeks to capture the other player E, the evader, according to stated rules. The game mentioned in the title is of this type, but it might be worth considering two much simpler examples first.

Damsel on lake. A young lady E, rowing on a lonely circular lake, discovers to her horror that the villain P is waiting to capture her when she lands on shore. Fleet of foot, she can outrun him on land but her rowing speed is only one quarter of his running speed. How can she escape? What she does is row in a circle concentric with the centre of the lake and of radius xr, where r is the radius of the lake. Provided that $x < \frac{1}{4}$, her angular velocity about the centre will exceed that of the villain and so eventually she will be able to place the centre between herself and the villain. At that point she rows directly for shore, a distance $(1-x)r$. The villain has to travel πr around the lake and so she escapes provided that $x > 1-\frac{1}{4}\pi$. Since the two conditions on x can just be met, she escapes.



Lion and gladiator.

A (hungry) lion P and a (presumably unarmed) gladiator E are placed in a circular arena with no exits. They can travel at exactly the same speed and are assumed to be able to manoevure at will. Can the gladiator escape the lion indefinitely? Surprisingly, he can, and one way is to follow a polygonal path consisting of a sequence of steps. In the nth of these, he starts off at a distance a_n from the centre of the arena (which we suppose to have radius r); he travels along a line at right angles to the diameter on which he lies, in a direction away from the semicircle containing the lion, and for a distance b_n.



Then $b_n^2 = (a_{n+1}^2 - a_n^2) > 2a_1(a_{n+1} - a_n)$.

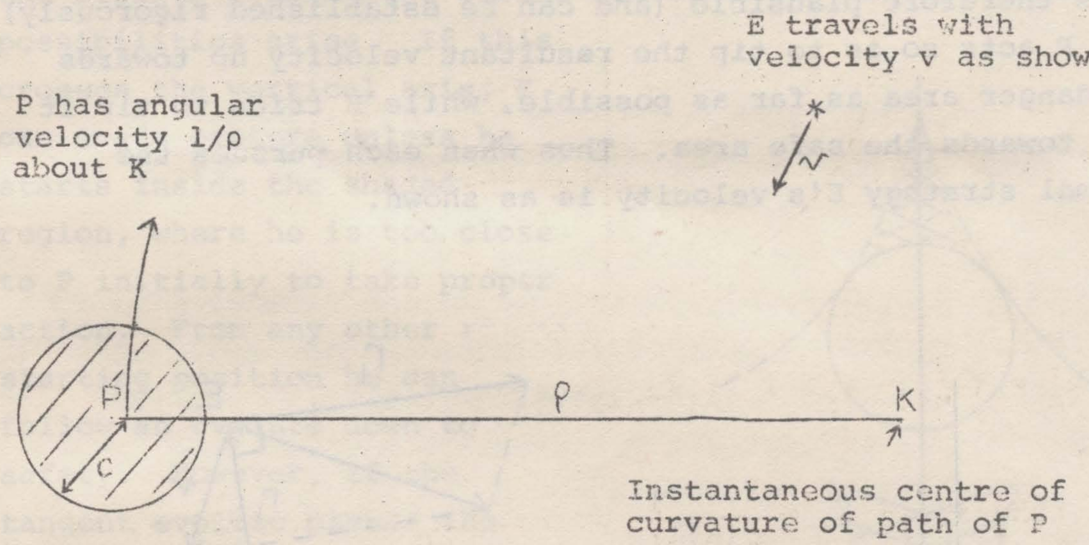
If we take $a_{n+1} - a_n = \frac{r - a_1}{n(n+1)}$ we have $a_n < r$ for all n, as required, and $b_n >$ constant $\times \frac{1}{n+1}$. Hence the total length of the gladiator's path is $\sum_{n=1}^{\infty} b_n$, which is infinite.

There is an entertaining account of this problem in "A mathematician's miscellany", by J. E. Littlewood, along with numerous other fascinating pieces of mathematics.

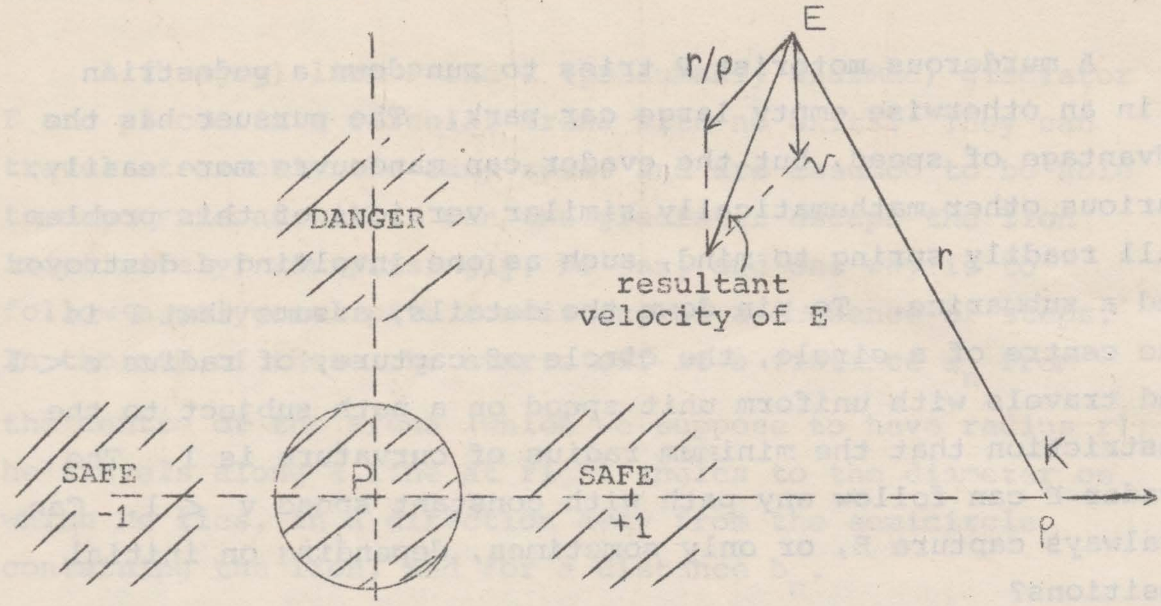
Homicidal chauffeur.

A murderous motorist P tries to run down a pedestrian E in an otherwise empty large car park. The pursuer has the advantage of speed, but the evader can manoeuvre more easily. Various other mathematically similar versions of this problem will readily spring to mind, such as one involving a destroyer and a submarine. To pin down the details, assume that P is the centre of a circle, the circle of capture, of radius $c < 1$, and travels with uniform unit speed on a path subject to the restriction that the minimum radius of curvature is 1. The evader E can follow any path with constant speed $v < 1$. Can P always capture E, or only sometimes, depending on initial positions?

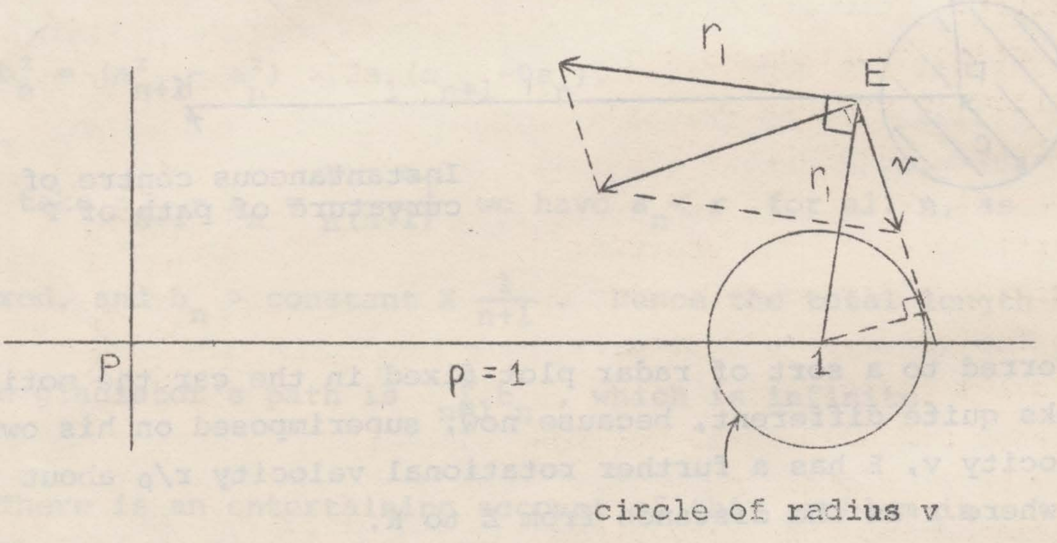
Suppose that at any instant the situation, as seen from above, is as shown.



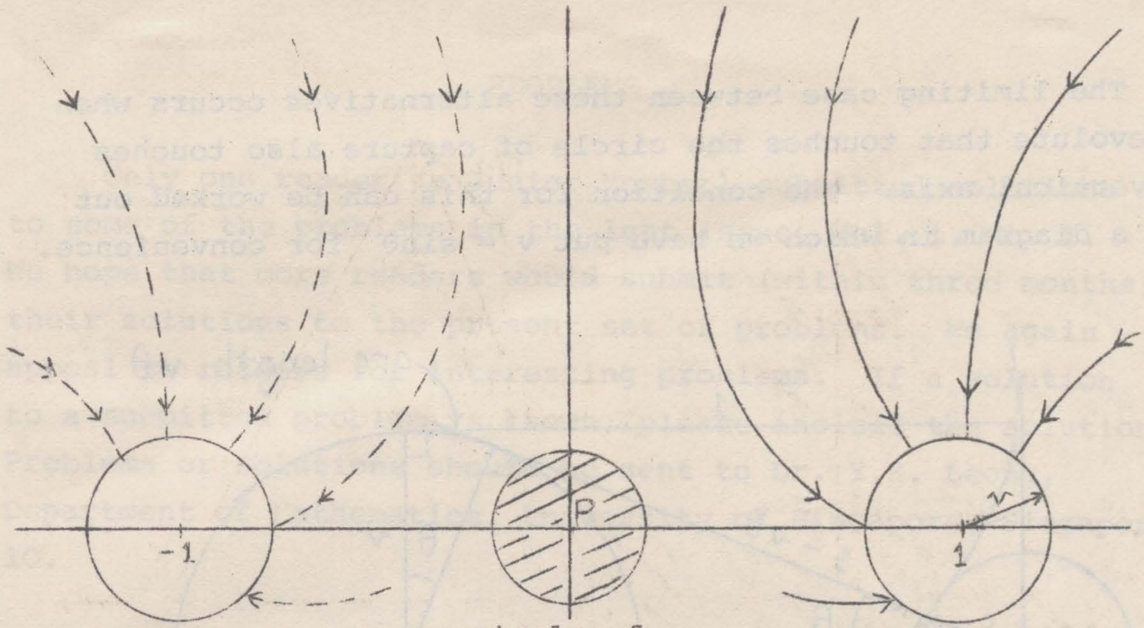
Referred to a sort of radar plot fixed in the car the motion looks quite different, because now, superimposed on his own velocity v, E has a further rotational velocity r/ρ about K, where r is the distance from E to K.



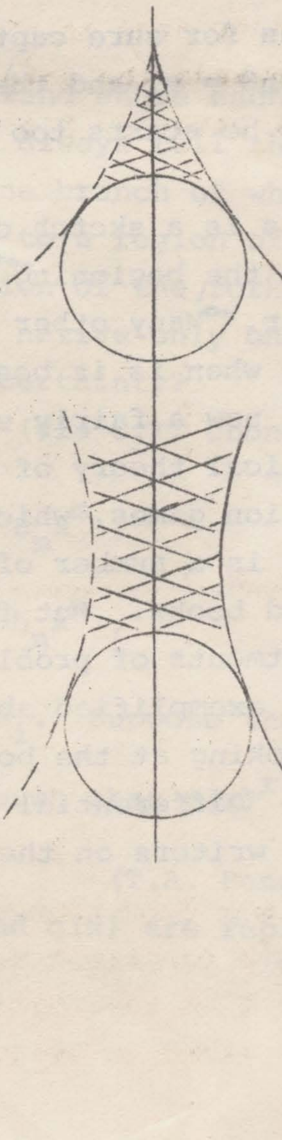
Notice that, on the radar plot, the area immediately ahead of P is the danger zone of imminent capture for E, while the areas around the extreme centres of curvature at $\rho = \pm 1$ are safety zones: P cannot turn sharply enough to reach E there. It is therefore plausible (and can be established rigorously) that P acts so as to tip the resultant velocity up towards the danger area as far as possible, while E tries to tip it down towards the safe area. Thus when each pursues the optimal strategy E's velocity is as shown.



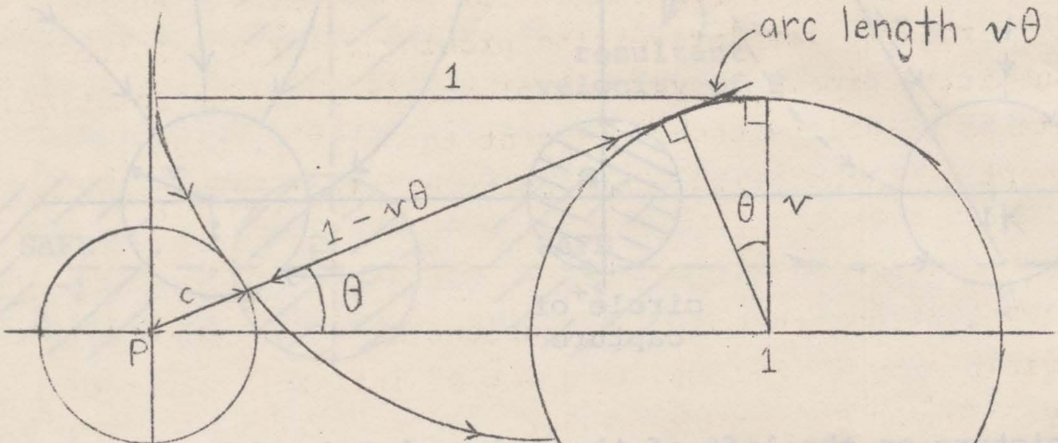
We now see from congruent triangles that (still in the radar plot) E's resultant velocity is at right angles to the tangent through E to the circle with centre ρ=1 and radius v. Thus E moves along an evolute to that circle.



The picture on the left of the radar plot is the mirror image of the right half. If E's path down an evolute takes him into the circle of capture, he loses; if it does not, he can evade capture. So the crucial question is what happens to the evolute grazing the circle of capture. Two alternative possibilities arise. If this crosses the vertical axis, E can evade capture unless he starts inside the shaded region, where he is too close to P initially to take proper action. From any other starting position he can follow an evolute down to safety. However, if the tangent evolute misses the vertical axis, the starting positions for capture open out to include all sufficiently distant points and P can always run down E (if necessary by retreating to a distance and then travelling towards him).



The limiting case between these alternatives occurs when the evolute that touches the circle of capture also touches the vertical axis. The condition for this can be worked out from a diagram in which we have put $v = \sin\theta$ for convenience.



$$\cos\theta = c + 1 - v\theta = c + 1 - \theta\sin\theta$$

Thus for sure capture, we need $c > \theta \sin\theta + \cos\theta - 1$, where $\sin\theta = v$, and the opposite inequality gives safety for E unless he starts too close to P.

This is a sketch only of the solution of the problem posed at the beginning of the discussion of the homicidal chauffeur. Many other related problems can be asked; for example, when is it best for P to start with a swerve? There is now a fairly extensive mathematical theory of pursuit and evasion games, which can be found in a number of recently published books. But for elementary treatments of problems of the sort exemplified above it is worth looking at the book entitled "Differential Games", by R. Isaacs, one of the earliest writers on the subject.