The proving of Cayley’s theorem at Further Mathematics level

Oey Liang Hien

Undoubtedly one of the hardest theorem to prove in "A" level syllabus of Further Mathematics is Cayley’s theorem in group theory: "Every finite group of order n is isomorphic to a permutation group on n symbols." This poses to teachers who must teach the theorem at pre-university level a challenge to make their students understand it and its proof on the spot without much difficulty.

In this article, I like to present a method of proof which I think is the most suitable one at this level. In choosing a suitable method of proof, I have been guided by the following points:

(1) Students are, in general, weak in abstract concepts.
(2) No use should be made of concepts which are not in the syllabus. (For example, the proof given by J. A. Green [1] is not suitable as it makes use of the homomorphism theorem.)
(3) It is preferable to have a long but intelligible proof rather than a short but difficult proof. Students are quite prepared to follow a long chain of carefully reasoned steps.

Thus, to overcome the abstractness of whatever proof we present, we could first make the proof more "concrete" by proving the theorem for one particular case before giving the general proof. I estimate that should I prove it generally at once, about 20 per cent of the students would understand it whereas if I prove a particular case and then generalize, 85 to 90 per cent would understand it.

This method of proof is given by Frank Ayres, Jr [2]. Moreover, in this proof there is only one step which is not easily understood by students: if \( G = \{ g_1, g_2, \ldots, g_n \} \) is a group with respect to the operation \( \ast \) and \( p_j \) is the permutation...
\[ P_j = \left( \begin{array}{cccc} g_1 & g_2 & \cdots & g_n \\ g_1 \ast g_j & g_2 \ast g_j & \cdots & g_n \ast g_j \end{array} \right), \]

where \( g_j \in G \), then \( P_j \) can also be written as
\[ P_j = \left( \begin{array}{cccc} g_1 \ast g_k & g_2 \ast g_k & \cdots & g_n \ast g_k \\ (g_1 \ast g_k) \ast g_j & (g_2 \ast g_k) \ast g_j & \cdots & (g_n \ast g_k) \ast g_j \end{array} \right) \]

for any given \( g_k \) in \( G \).

To convince students, we first give an example to show that in the expression of a permutation, the ordering of the columns is not important:
\[
\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 4 & 1 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 3 & 2 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix}, \text{ etc.}
\]

Next, we point out that the columns and rows of a group multiplication table satisfy the Latin square property. That is, each row (or column) contains all the elements of the group without repetition:

\[
\begin{array}{c|cccccccccc}
\ast & g_1 & g_2 & \cdots & g_k & \cdots & g_n \\
\hline
\quad g_1 & g_1 \ast g_k \\
\quad g_2 & g_2 \ast g_k \\
\quad \vdots & \quad \vdots \\
\quad g_n & g_n \ast g_k \\
\end{array}
\]

These two observations will convince the students of the validity of the second expression for \( P_j \).

Let us verify Cayley's theorem in the case of a particular group \( G \) with the following multiplication table:
Using the previous notations, define

\[
\begin{align*}
P_1 &= (1) \quad , \quad P_2 = (12)(36)(45) , \\
P_3 &= (13)(25)(46) , \quad P_4 = (14)(26)(35) , \\
P_5 &= (156)(234) , \quad P_6 = (165)(243) .
\end{align*}
\]

Thus we have the permutations

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Form the multiplication table of the set \(P = \{ P_1, P_2, P_3, P_4, P_5, P_6 \} \) under composition of permutations.
Thus $p$ forms a group under composition.

Define the mapping $f$ from $G$ to $p$ by

$$f(i) = p_i, \quad i = 1, 2, \ldots, 6.$$  

It can be easily seen from the table that $f$ is one-to-one and onto, and also preserves the binary operations. Hence $G$ is isomorphic to $p$.

We now proceed to give the general proof of Cayley's theorem.

Let $G = \{g_1, g_2, \ldots, g_n\}$ be a group under the operation $\cdot$.

For each $i = 1, 2, \ldots, n$, define the permutation $p_j$ which we simply write as

$$p_j = (g_{i_1}, g_{i_2}, \ldots, g_{i_n})$$

where $i_1, i_2, \ldots, i_n$ is the set of elements in the second row in $i$-th column of the multiplication table of $G$. We now show that the set

$$\{p_1, p_2, \ldots, p_n\}$$

forms a group under composition.

Note that $p_j$ is a permutation on the $n$ elements of $G$ since the elements in the second row occur in one column of the multiplication table of $G$ and hence run through all the elements of $G$. We now show that the set

$$\{p_1, p_2, \ldots, p_n\}$$

forms a group under composition.
is a permutation group on $n$ symbols.

Let $P_j, P_k \in G$. Then

$$P_j \circ P_k = \begin{pmatrix} g_1 \\ g_i \end{pmatrix} \circ \begin{pmatrix} g_i \\ g_{i\ast g_k} \\ g_{i\ast g_j} \\ (g_i \ast g_j) \ast g_k \end{pmatrix} = \begin{pmatrix} g_i \ast g_j \\ g_1 \ast (g_j \ast g_k) \end{pmatrix},$$

by the remark mentioned earlier.

It can be easily seen from this that it is one-to-one.

Hence $P$ is isomorphic to $G$.

Now, we are to prove the set $P$ is a group.

To a group we must define an operation such that

- composition is associative
- the identity element is $P_0 = \begin{pmatrix} \epsilon_g \\ \epsilon_g \end{pmatrix}$
- for every $P = \begin{pmatrix} g_i \\ g_i \end{pmatrix}$ there exists an inverse

Composition is associative and the identity element is $\begin{pmatrix} \epsilon_g \\ \epsilon_g \end{pmatrix}$. The inverse of $\begin{pmatrix} g_i \\ g_i \end{pmatrix}$ is $\begin{pmatrix} g_i \\ g_i^{-1} \end{pmatrix}$. 

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Similarly, the winning solution will be decided by adding the scores of the two players. A winning solution is one where the player with the highest score wins.

Thus $P$ is a group under the operation of composition.

Define the mapping $\alpha$ from $G$ to $P$ such that

\[ \alpha(g_i) = p_i, \quad i = 1, 2, \ldots, n. \]

Moreover, $\alpha$ is clearly one-to-one. To show that $\alpha$ preserves the group operations, we have

\[ \alpha(g_i \cdot g_j) = \alpha(g_i) \cdot \alpha(g_j) = (p_i \cdot p_j) = (g_i \cdot g_j). \]

Hence $\alpha$ is an isomorphism from $G$ onto $P$.

I would like to mention that I tried this method on
my students at the National Junior College with satisfying results.

References


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Research Fellowships in Germany

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