PROBLEMS AND SOLUTIONS

A book-voucher prize will be awarded to the best solution of a starred problem. Only solutions from Junior members and received before 1 July, 1977 will be considered for the prizes. If equally good solutions are received, the prize or prizes will be awarded to the solution or solutions sent with the earliest postmark. In the case of identical postmarks, the winning solution will be decided by ballot.

Problems or solutions should be sent to Dr. K. Y. Woo, Department of Mathematics, University of Singapore, Singapore 10. Whenever possible, please submit a problem together with its solution.

*P1/77 Prove, preferably by induction, that for $n > 1$,

$$n^n > (n+1)^{n-1}$$

(via Chan Sing Chun)

*P2/77 Prove that for any positive integer n,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^n = n!$$

(via Chan Sing Chun)

(This is taken from the 1976 Putnam Competition.)

*P3/77 Two positive integers are said to be relatively prime if their only common factor is 1. What is the probability that two numbers selected at random are relatively prime? (You may assume that the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ converges to $\frac{\pi^2}{6}$

and that $\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right)\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\cdot$

$\left(1 - \frac{1}{7^2}\right) \ldots = 1.$)

(via L. Y. Lam)

*P4/77 Five sailors plan to divide a pile of coconuts among themselves in the morning. During the night one of then wakes up and decides to take his share. After throwing a coconut to a monkey to make the division come out even, he takes one fifth of the pile and goes back to sleep. The other four sailors do likewise, one after the other, each throwing a coconut to the monkey and taking one fifth of the remaining pile. In the morning the five sailors throw a coconut to the monkey and divide the remaining coconuts into five equal piles. What is the smallest number of coconuts that could have been in the original pile?

(via K. Y. Woo)

Solutions to P9 - P12/76:

*P9/76 Let P be a probability. Prove that for any two events A and B

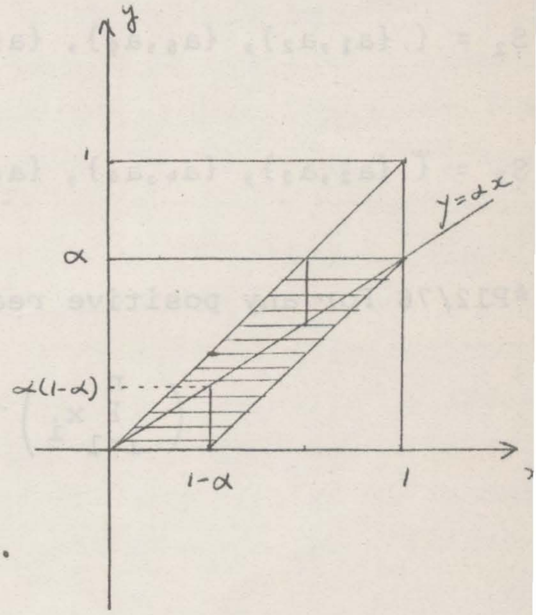
$$|P(A\cap B) - P(A)\cdot P(B)| \leq 0.25.$$

Give an example to show that the upper bound of 0.25 may be attained.

(via Louis H. Y. Chen)

Solution by Tay Yong Chiang:

Let $P(A) = x$, $P(B) = \alpha$. Then $P(A\cap B)$ is constrained by $P(A\cap B) \leq \alpha$, $P(A\cap B) \leq x$, and since $P(A\cup B) = x + \alpha - P(A\cap B)$, $0 \leq x + \alpha - P(A\cap B) \leq 1 \Rightarrow$ $P(A\cap B) \geq x + (\alpha-1)$, $P(A\cap B) \leq x + \alpha$ (Editor's note: The second inequality should be replaced by $P(A\cap B) \leq x$.) Let $y = P(A\cap B)$. Then the ordered pair (x,y) is restricted to the shaded region in the graph shown. Consider, for any x, the difference between $y = P(A\cap B)$ and $y = \alpha x$. From the graph, $|P(A\cap B) - \alpha x| \leq \alpha(1-\alpha)$.



'Completing the squares' gives $\alpha(1-\alpha) \leq 0.25$. Hence $|P(A\cap B) - P(A)P(B)| \leq 0.25$. Example when upper bound is attained: $P(A) = P(B) = 0.5$, $P(A\cap B) = 0$ or 0.5.

Joseph Lee and Chan Sing Chun gave an explicit example: A fair coin is thrown. A is the event "the coin shows head". B is the event "the coin shows tail". $P(A) = P(B) = \frac{1}{2}$. $P(A \cap B) = 0$. Therefore $|P(A \cap B) - P(A) \cdot P(B)| = \frac{1}{4}$.

(Also solved by Proposer)

*P10/76

An incomplete solution was received.

P11/76 Let $a_0, a_1, \ldots, a_6, b_0, b_1, \ldots, b_6$ be distinct elements. Let $S = \{ \{a_0,a_1\}, \{a_1,a_2\}, \{a_2,a_3\}, \{a_3,a_4\}, \{a_4,a_5\}, \{a_5,a_6\}, \{a_6,a_0\}, \{a_0,b_0\}, \{a_1,b_1\}, \{a_2,b_2\}, \{a_3,b_3\}, \{a_4,b_4\}, \{a_5,b_5\}, \{a_6,b_6\}, \{b_0,b_2\}, \{b_2,b_4\}, \{b_4,b_6\}, \{b_6,b_1\}, \{b_1,b_3\}, \{b_3,b_5\}, \{b_5,b_0\} \}$. Find subsets S_1, S_2, S_3 such that S is the disjoint union of S_1, S_2, S_3 and each of the a_i and b_i appears at most once in each of S_1, S_2, S_3.

(via H. P. Yap)

Solution by Tay Yong Chiang:

There is no unique solution. Consider, for example,

$$S_1 = \{ \{a_0,a_1\}, \{a_3,a_4\}, \{a_5,a_6\}, \{a_2,b_2\}, \{b_4,b_6\}, \{b_1,b_3\}, \{b_5,b_0\} \}$$

$$S_2 = \{ \{a_1,a_2\}, \{a_6,a_0\}, \{a_3,b_3\}, \{a_4,b_4\}, \{a_5,b_5\}, \{b_0,b_2\}, \{b_6,b_1\} \}$$

$$S_3 = \{ \{a_2,a_3\}, \{a_4,a_5\}, \{a_0,b_0\}, \{a_1,b_1\}, \{a_6,b_6\}, \{b_2,b_4\}, \{b_3,b_5\} \}.$$

*P12/76 For any positive real numbers $x_1, x_2, \ldots, x_n$, prove that

$$\left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} x_i^3 \right) \geqslant \left(\sum_{i=1}^{n} x_i^2 \right)^2 .$$

(via H. N. Ng)

Solution by Chan Sing Chun:

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i^3\right) - \left(\sum_{i=1}^{n} x_i^2\right)^2$$

$$= \sum_{i=2}^{n} x_1x_i(x_1-x_i)^2 + \sum_{i=3}^{n} x_2x_i(x_2-x_i)^2 + \sum_{i=4}^{n} x_3x_i(x_3-x_i)^2 + \ldots + x_{n-1}x_n(x_{n-1}-x_n)^2 \geqslant 0 .$$

(Also solved by Y. K. Leong, Tay Yong Chiang)

Tay Yong Chiang has been awarded the prize for correct solution to P9/76.

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