A sum of the form

\[ \sum_{r=1}^{n} f(x_{r+1}, x_{r+2}, \ldots, x_{r+n}), \]

where \( x_{s+n} = x_s \) for each \( s \) and \( n \) is a positive integer, is called a cyclic sum. If this sum is denoted by \( F_n(x_1, x_2, \ldots, x_n) \) then it is clear that

\[ F_n(x_{s+1}, x_{s+2}, \ldots, x_{s+n}) = F_n(x_1, x_2, \ldots, x_n) \]

for each \( s \). It is because of this that the sum is called a cyclic sum.

In this paper we are concerned mainly with inequalities for the cyclic sum

\[ S_n(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{n} \frac{x_r}{x_{r+1} + x_{r+2}}, \]

where \( x_{s+n} = x_s \), \( x_{s+1} + x_{s+2} > 0 \) for each \( s \).

It is trivial that \( S_1(x_1) = \frac{1}{2} \) and \( S_2(x_1, x_2) = 1 \).

In 1903, A. M. Nesbitt [1] asked for a proof of the inequality

\[ S_3(x_1, x_2, x_3) \geq \frac{3}{2}. \]

(Three known proofs of this are given in the appendix to this paper).

Over 50 years later, in 1954, H. S. Shapiro [2] asked for a proof of the inequality

\[ S_n(x_1, x_2, \ldots, x_n) \geq \frac{n}{2} \]
for all positive integers $n$. At the present time it is known that this inequality is true for all $n \leq 10$ and false for all even $n > 14$ and all odd $n > 25$. For each of the remaining values of $n$ (namely 11, 12, 13, 15, 17, 19, 21 and 23) it is not known whether the inequality is true or false.


M. J. Lighthill (see [9]) proved that the inequality is false for $n = 20$. He himself extended his method to prove the inequality false for $n = 14$. His proof has not been published. For $n = 14$, using Lighthill's method, A. Zulauff [10] and M. Herschorn and J. E. L. Peck [11] proved the same result. That the inequality is false for all even $n > 14$ follows from this, since it can be easily seen that

$$S_{n+2}(x_1, x_2, \ldots, x_n, x_1, x_2) = S_n(x_1, x_2, \ldots, x_n) + 1.$$ 

R. A. Rankin [12], using Lighthill's result for $n = 20$, proved that the inequality is false for all sufficiently large odd $n$. Later A. Zulauff [13] proved that the inequality is false for all odd $n > 53$. This was improved by the author [7], who proved that the inequality is false for all odd $n > 27$. Later, D. E. Daykin [14] and M. A. Malcolm [15] proved that the inequality is false for $n = 25$.

In his paper [12], Rankin also proved that there is a positive number $\lambda < \frac{1}{4}$ with the property that
is true for all \( n \), but

\[ S_n(x_1, x_2, \ldots, x_n) \not\geq (\lambda + \varepsilon)n \]

is not true for all \( n \) and all \( x_1, x_2, \ldots, x_n \), however small \( \varepsilon > 0 \) is. He stated that he could prove that \( \lambda > 0.3 \). Later he published a proof showing that \( \lambda > 0.33 \). The author proved that \( \lambda > 0.45 \) in [17] and later that \( \lambda > 0.46 \) in [18]. More recently V. G. Drinfeld [19] has proved that \( \lambda = 0.494 \ldots \).

In [14] and [20], the author investigated the inequality

\[ T_n(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{n} \frac{x_r}{x_{r+1} + x_{r+2} + \ldots + x_{r+m}} \geq \frac{n}{m}, \]

where \( x_{n+s} = x_s > 0 \) and \( x_{s+1} + x_{s+2} + \ldots + x_{s+m} > 0 \) for each \( s \), and proved that the inequality is true if

- \( n \mid m+2 \) or \( 2m \) or \( 2m+1 \) or \( 2m+2 \),
- \( n \mid m+3 \) and \( n = 8 \) or \( 9 \) or \( 12 \),
- \( n \mid m+4 \) and \( n = 12 \).

For \( m \geq 3 \) it is not known whether there are any other \((m, n)\) for which the inequality holds.

D. E. Daykin [14] considered the inequality

\[ \sum_{r=1}^{n} \left( \frac{2x_r}{x_{r+1} + x_{r+2}} \right)^t \geq n, \]

where \( x_{s+n} = x_s > 0 \) and \( x_{s+1} + x_{s+2} > 0 \) for each \( s \), and proved that it is true for \( t \geq 2 \). Using his method, the author [21] proved that the inequality is true for \( t \geq \frac{\sqrt{5}+1}{2} \).
The smallest $T$ such that the inequality is true for all $n$ and all $t \geq T$ is not known.

Other related inequalities have also been studied by various authors (see, e.g., [3], [4], [14], [17], [18] and [20] to [24]). An expository account of cyclic inequalities, covering many of the publications up to 1968, is given in the book [25] by D. S. Mitronović.

References

3. Mordell, L. J. On the inequality $\sum_{r=1}^{n} \frac{x_r}{(x_{r+1}+x_{r+2})} \geq n/2$ and some others. Abh. Math. Univ. Hamburg 22 (1958), 229.
6. Bajsanski, B. A remark concerning the lower bound of $x_1/(x_2+x_3) + x_2/(x_3+x_4) + \ldots + x_n/(x_1+x_2)$. Univ. Beograd Publ. Electrotehn. Fac. Ser. Mat. Fiz. No. 70-76 (1962), 19.
Appendix

Below are three proofs of the inequality

\[ S_3(x_1, x_2, x_3) \geq \frac{3}{2}, \]

**First proof.** The inequality is equivalent to

\[
\frac{x_1+x_2}{x_2+x_3} + \frac{x_2+x_3}{x_3+x_1} + \frac{x_3+x_1}{x_1+x_2} \geq 6,
\]

which follows from the inequality between arithmetic and geometric means.

This proof can be generalized to prove that

\[ S_n(x_1, x_2, \ldots, x_n) \geq \frac{n}{2}, \]

**Second proof.** The inequality is equivalent to

\[
(x_1+x_2)(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}) \geq \frac{9}{2},
\]

which is true, since

\[
\frac{1}{3}(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}) \geq \frac{3}{2(x_1+x_2+x_3)}
\]

by the inequality between arithmetic and harmonic means.

This proof can be generalized to prove that

\[ T_{m+1}(x_1, x_2, \ldots, x_{m+1}) \geq \frac{m+1}{m}. \]

**Third proof.** Considering \( S_3(x_1, x_2, x_3) \) as a weighted sum of \( \frac{1}{x_2+x_3}, \frac{1}{x_3+x_1}, \frac{1}{x_1+x_2} \) with weights \( x_1, x_2, x_3 \), respectively, we can see that

\[
S_3(x_1, x_2, x_3) \geq \frac{(x_1+x_2+x_3)^2}{x_1(x_2+x_3)+x_2(x_3+x_1)+x_3(x_1+x_2)}
\]
by the inequality between weighted arithmetic and harmonic means. Hence the inequality \( S_3 \geq 3/2 \) follows if we can prove that the quadratic form

\[
(x_1 + x_2 + x_3)^2 - \frac{3}{2} x_1 (x_2 + x_3) - \frac{3}{2} x_2 (x_3 + x_1) - \frac{3}{2} x_3 (x_1 + x_2)
\]

is positive semi-definite. This is true since the quadratic form is equal to

\[
(x_1 - \frac{1}{2} x_2 - \frac{1}{2} x_3)^2 + \frac{1}{2} (x_2 - x_3)^2.
\]

This last proof is more complicated than either of the other two proofs. It can, however, be generalized to prove that

\[
S_n(x_1, x_2, \ldots, x_n) \geq \frac{n}{2}
\]

for \( n = 4, 5, 6 \) (see [4]), and also to prove all the known true cases of

\[
T_n(x_1, x_2, \ldots, x_n) \geq \frac{n}{m}
\]

for \( m > 2 \) (see [4], [20]).