ON THE TRISECTION OF AN ANGLE

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Every now and then we receive claims that a construction for trisecting an angle using ruler and compasses has been found. We know very well that there must be fallacies hidden in such a construction as it has been proved more than a century ago that the problem of trisecting an angle is unsolvable. The fallacies may be that either (a) additional instruments are used knowingly or unknowingly, or (b) the construction gives only an approximate solution or solution to special and not general angle.

When teaching Galois theory last semester and admiring its intrinsic beauty, I found a class of angles that cannot be trisected, namely the acute angles of a right-angled triangle two of whose sides have an integral ratio. I present these angles together with proof in the hope that it would reassure would-be proposers of constructions for trisecting an angle that the problem is actually unsolvable.

For the benefit of those readers who are not too familiar with the relevant portion of algebra, we recapitulate necessary results on geometrical constructibility in the preliminary section. This is to be followed by our main results on the constructibility of an angle whose cosine (or tangent) is a rational number.

It should perhaps be mentioned that while trisection of an arbitrary angle by (unmarked) ruler and compasses is impossible, there do exist in the literature methods of trisection of an angle with <u>additional aids</u>. To quote a few :

A) Constructions by reduction to certain "vergings"

a) Pappus' construction [3, pp.235-236]

b) Archimedes' construction [3, p.241]

c) Euclid's construction [2, pp. 285-286]

B) Direct constructions by means of conics

a) Pappus' first construction [2, pp.205-206]

b) Pappus' second construction [2, pp.206-207]

c) Descartes' construction [4, pp.53-54]

C) Others

a) Construction by means of Hippias' trisectrix [2, pp.75-76]

b) Kopf's approximate construction [4, pp.54-55].

1. Constructibility

The geometers of ancient Greece posed in the fifth century B.C. the following three elementary problems that were to fascinate professional and amateur mathematicians in antiquity as well as in modern times and to defy their ingenuity for many years:

Problem 1. (*Trisecting an angle*) *To divide an angle into three equal parts.*

Problem 2. (*Doubling a cube*) *To construct the side of a cube whose volume is twice that of a given cube.*

Problem 3. (*Squaring a circle*) *To construct the side of a square whose area equals to that of a given circle.*

It was proved more than 2200 years later that all three of the problems were unsolvable by means of (unmarked) ruler and compass alone. Although none of the three problems is of mathematical importance once they were resolved, they do have their historical significance. Problem 3 necessitated a widening and sharpening of the number concept while the historical importance of Problems 1 and 2 is the impetus they gave to the investigation of the arithmetic nature of the roots of algebraic equations, culminating in the modern concepts of groups and fields.

We shall outline a proof of the unsolvability of the angle-trisection problem.

In a problem of geometrical construction, there is usually given a set of geometrical elements and we are required to produce a particular geometrical element from the given set by specified means. An analytical approach to the problem is to characterize each geometrical element by a number, or an ordered pair of numbers, or by a higher dimensional ordered tuple of numbers. For example, in the plane, a line segment can be characterized by its length, a point by its coordinates in a rectangular coordinate system, an angle between 0° and 180° by its cosine, a line by its angle of inclination to the x-axis or by two points (as the case may be), a circle by its center and radius, etc.

To be given a set of geometrical elements is essentially to be given a set of numbers. The construction of the required geometrical element can be accomplished by constructing the numbers characterizing the geometrical element. Analytically, it is usually possible to find the relation, in the form of an equation, between the required numbers and the given numbers. The problem then reduces to the construction of roots of the resulting equation. For example, in Problem 1, we are given the cosine, say c, of an angle (which can be assumed to be acute) and we are required to find the cosine, say x, of a third of the given angle. In essence, trisection of an angle is equivalent to constructing the roots of the equation

$$4x^3 - 3x = c \quad (0 < c < 1).$$

Likewise, Problem 2 asks for a construction of a root of the equation

$$x^3 = 2.$$

Let us now examine what sort of numbers can be constructed out of a given set of numbers by ruler and compasses. A real number α is constructible if one can construct a line segment of length $|\alpha|$ in a finite number of steps from the given line segments representing the given numbers. A complex number is constructible if both its real and imaginary parts are constructible.

<u>Theorem 1</u>. *Given a set of numbers which include 1. Then*

(a) *if α and β are constructible and $\beta \neq 0$, then so are $\alpha + \beta$, $\alpha - \beta$, $\alpha\beta$, α/β and $\sqrt{\beta}$. (Addition, subtraction, multiplication and division by nonzero numbers are referred to as rational operations.)*

(b) *the set of constructible numbers consists precisely of all those that can be obtained from the given numbers by a finite number of rational operations and the extractions of square roots.*

A proof of the above theorem can be found in Bold [1, pp.1-2, 7-9] .

A set of numbers which is closed under rational operations is called, in abstract algebra, a field. Given a set S of numbers, we shall denote the field of all numbers that can be obtained from S by a finite number of rational operations by K_o. For any α in K_o with $\sqrt{\alpha}$ not in K_o, the set of all elements $a + b\sqrt{\alpha}$, where a and b belong to K_o, is a field and we shall denote this field by $K_o(\sqrt{\alpha})$. For β in $K_o(\sqrt{\alpha})$ and $\sqrt{\beta}$ not in $K_o(\sqrt{\alpha})$, we denote the field of all elements $c + d\sqrt{\beta}$ (c, d in $K_o(\sqrt{\alpha})$) by $K_o(\sqrt{\alpha},\sqrt{\beta})$. We may proceed in this way to define bigger fields. It is obvious that a number ω is constructible if and only if there exists a sequence of numbers : $\alpha, \beta, \ldots, \gamma, \delta$ where α belongs to K_o, β to $K_o(\sqrt{\alpha})$, ... , δ to $K_o(\sqrt{\alpha},\sqrt{\beta},..,\sqrt{\gamma})$ and ω to $K_o(\sqrt{\alpha},\sqrt{\beta},\ldots,\sqrt{\gamma},\sqrt{\delta})$.

As we have noted earlier the problem of trisecting an angle with cosine c is essentially equivalent to that of constructing the roots of the equation $4x^3 - 3x = c$. Regarding the constructibility of the roots of a cubic equation, we have the following

Theorem 2. *Assume that $x^3 + px^2 + qx + r = 0$ is a cubic equation with coefficients in K_o. Then*

(a) all roots of the equation are constructible if the equation has a root in K_o,

(b) none of the roots of the equation can be constructed if the equation has no roots in K_o.

For the proof, we refer the reader to Bold [1,pp.13-16].

If there is a general method to trisect an arbitrary angle, then it can be applied to trisect an angle of 60^0. Since $\cos 60^0 = 1/2$, K_o is the field of all rational numbers. The equation $4x^3 - 3x = 1/2$ has no rational roots. By Theorem 2, $\cos 20^0$ is constructible. Thus the general method fails to be applicable in this particular case.

2. Trisection of an angle whose cosine is a rational number

Let θ be an angle ($0 < \theta < 180^0$) whose cosine is a rational number m/n (n > 0, m and n are relatively prime.) Our given set of numbers consists of 1 and m/n, and the field K_o is just the field of all rational numbers. By Theorem 2, θ can be trisected if and only if the following equation

(1) $$4x^3 - 3x = m/n$$

has a rational root. If α/β ($\beta > 0$, α and β are relatively prime) is a rational root of Eq.(1), then

$$4(\alpha/\beta)^3 - 3(\alpha/\beta) = m/n$$

or

(2) $$\frac{m}{n} = \frac{\alpha(4\alpha^2 - 3\beta^2)}{\beta^3}.$$

This prove half of the following lemma :

Lemma 1. *An angle θ ($0 < \theta < 180^0$) whose cosine is a rational number m/n can be trisected if and only if m/n is of the form (2) where α and β are relatively prime integers such that $\beta > 0$ and $-\beta < \alpha < \beta$.*

Proof. We have shown that if θ can be trisected, then m/n is of the required form (2).

Conversely, if m/n is of the form (2), then α/β is a root of Eq.(1). By Theorem 2, all the roots of the equation can be constructed. In particular, the angle which is a third of θ can be constructed.

Remark 1. If α/β is a root of the equation

$$4x^3 - 3x = \alpha(4\alpha^2 - 3\beta^2)/\beta^3 ,$$

then the other two roots are

$$\frac{1}{2\beta}(-\alpha \pm \sqrt{3\beta^2 - 3\alpha^2}).$$

When α/β is denoted by $\cos\phi$, then these two roots can be expressed as $\cos(\phi \pm 120^0)$. The three angles ϕ, $\phi + 120^0$ and $\phi - 120^0$ are a third of the angles θ, $360^0 + \theta$ and $720^0 + \theta$ (not necessarily in that order).

Remark 2. There are infinitely many rational numbers of the form $\alpha(4\alpha^2 - 3\beta^2)/\beta^3$, where α and β are relatively prime integers and $|\alpha| < \beta$. Hence there are infinitely many angles which can be trisected.

Theorem 3. *An angle whose cosine is 1/n, where n is a positive integer greater than 1, cannot be trisected.*

Proof. Suppose on the contrary that there is an integer $n(\geqslant 2)$ such that an angle with cosine 1/n can be trisected. Then by Lemma 1, there are suitable integers α, β such that

$$1/n = \alpha(4\alpha^2 - 3\beta^2)/\beta^3 .$$

We will show that this is not possible.

We have

$$\beta^3 = n\alpha(4\alpha^2 - 3\beta^2) .$$

Hence α divides β^3, and so $\alpha = \pm 1$ since α and β are relatively prime. If $\alpha = 1$, we have

$$\beta^3 = n(4 - 3\beta^2) .$$

As $\beta > 0$, the only possibility is $\beta = 1$, and then $n = 1$, a contradiction. Finally, if $\alpha = -1$, we have

$$\frac{4n}{\beta^2} = 3n - \beta .$$

We consider two cases : (i) $\beta < 2n$, (ii) $\beta \geqslant 2n$. In case (i), we would have $\beta^2 < 4$ so that $\beta = 1$. This implies that $n = -1$, a contradiction. Lastly, in case (ii), $4n/\beta^2 \leq 1/n \leq 1$, contradicting the assumption that $3n - \beta$ is an integer.

Remark 3. By a similar argument, we can show that an angle whose cosine is $-1/n$ $(n > 2)$ cannot be trisected.

3. Trisection of an angle whose tangent is a rational number

Lemma 2. *An angle θ $(0 < \theta < \frac{1}{2}\pi)$ whose tangent is a rational number m/n can be trisected if and only if*

$$\frac{m}{n} = \frac{\alpha(\alpha^2 - 3\beta^2)}{\beta(3\alpha^2 - \beta^2)}$$

for some integers α and β with $\beta > 0$, and α relatively prime to β .

Proof. If θ can be trisected, then the following equation

$$nx^3 - 3mx^2 - 3nx + m = 0 \tag{4}$$

has a rational root, α/β say ($\beta > 0$ and α, β relatively prime). Here we have made use of the triple-angle formula $\tan 3\phi = (3 \tan \phi - \tan^3 \phi)/(1 - 3 \tan^2 \phi)$. It follows that

$$\frac{m}{n} = \frac{3(\alpha/\beta) - (\alpha/\beta)^3}{1 - 3(\alpha/\beta)^2} = \frac{\alpha(3\beta^2 - \alpha^2)}{\beta(\beta^2 - 3\alpha^2)}.$$

Conversely, if m/n is of the form (3), then α/β is a root of (4). By Theorem 2, all roots of (4) can be constructed.

Remark 4. If α/β is a root of $(3\alpha^2 - \beta^2)\beta x^3 - 3(\alpha^2 - 3\beta^2)\alpha x^2 - 3(3\alpha^2 - \beta^2)\beta x + (\alpha^2 - 3\beta^2)\alpha = 0$, then the other roots are

$$(4\alpha\beta \pm (\alpha^2 + \beta^2)\sqrt{3})/(-3\alpha^2 + \beta^2),$$

or

$$(\alpha \pm \sqrt{3}\beta)/(\beta \mp \sqrt{3}\alpha).$$

If we denote α/β by $\tan \phi$, then the other roots are $\tan(\phi \pm 120^0)$.

Theorem 4. *An angle θ whose tangent is an integer n greater than 1 cannot be trisected.*

Proof. Suppose on the contrary that θ can be trisected. Then by Lemma 2,

$$n = \frac{\alpha(\alpha^2 - 3\beta^2)}{\beta(3\alpha^2 - \beta^2)}$$

for some integers α, β which are relatively prime and $\beta > 0$. Thus

$$n\beta(3\alpha^2 - \beta^2) = \alpha(\alpha^2 - 3\beta^2).$$

Since α, β are relatively prime, it follows that α divides n. Writing $n = \alpha k$, we have

$$\alpha^2(3k\beta - 1) = \beta^2(k\beta - 3). \tag{5}$$

Since β, $3k\beta - 1$ are relatively prime, β divides α^2, and hence $\beta = 1$. Substituting into (5), we have

$$\alpha^2 = (k - 3)/(3k - 1).$$

Now it is easily checked that $(k - 3)/(3k - 1) < 1$ if $k > \frac{1}{3}$ or $k < -1$. Therefore either $k = 0$ or -1. The case $k = 0$ implies that $\alpha^2 = 3$, which is impossible. Finally, the case $k = -1$ implies that $\alpha = \pm 1$, so that $n = \pm 1$, again a contradiction. Hence θ cannot be trisected.

Remark 5. By a similar argument, we can show that an angle whose tangent is $\pm 1/n$ or $-n$, where n is an integer greater than 1, cannot be trisected.

Remark 6. Theorem 3 and 4 imply that in a right-angled triangle in which the ratio of any two of its sides is an integer greater than 1, the two acute angles of the triangle cannot be trisected.

References

1. Benjamin Bold, *Famous problems of mathematics*, Van Nostrand Reinhold, New York, 1969.

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3. Thomas Heath, *A history of Greek mathematics*, Vol. I, Oxford University Press, London, 1921.

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