# TEACHING NOTES

## A NOTE ON IDENTITY ELEMENTS

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In most books on elementary group theory, the axiom on the existence of an identity element e in a non-empty set G with a binary operation $*$ is simply mentioned as:

for all $a \in G$, $\quad e * a = a = a * e$ ..............(I)

without further elaboration. As a result, students very ofter fail to gain an insight into the meaning of (I). The two common mistakes they make are: Firstly, they fail to see that the condition

$$e * a = a = a * e$$

must hold for all $a \in G$. Secondly, they tend to prove either $e * a = a$ or $a * e = a$ only.

We hope that this note will help students have a better understanding of the identity axiom. Also, we show how, given a set with a binary operation defined on it, one may find the identity element. Further, we hope that students will be able to define new operations using our techniques.

Let $*$ be a binary operation on $\mathbb{R}$, the set of real numbers, defined by $a * b = a + (b - 1)(b - 2)$. We note that for all $a \in \mathbb{R}$,

$$a * 1 = a$$

and

$$a * 2 = a.$$

The numbers 1 and 2 are known as right identity elements with respect to the operation $*$.

On the other hand, there exists no element r (say) such that $r * b = b$ for all $b \in \mathbb{R}$. This can be proved by contradiction as follows. If possible, $r * b = b$ for all $b \in \mathbb{R}$. Hence

$$r + (b - 1)(b - 2) \equiv b$$

which is an identity and such an identity is impossible.

We make the following definition. Let $*$ be a binary operation defined on a non-empty set A. An element $u \in A$ such that $a * u = a$ for all $a \in A$ is said to be a right identity element with respect to $*$. Similarly, an element $v$ is a left identity element if $v * a = a$ for all $a \in A$.

For example, the operation $o$ on $\mathbb{R}$ defined by

$$a \, o \, b = a(a^2 - 1) + b$$

has three left identity elements 0, 1 and -1, but there exists no right identity element.

It is easy to see that the binary operation # on $\mathbb{R}$ defined by $a \# b = ab + 1$ has neither right identity nor left identity elements.

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In the following, we shall generalise the results pertaining to the above binary operations.

Let $*$ be a binary operation on $\mathbb{R}$ expressible in the form

$$a * b = a + g(a)f(b)$$

where f and g are real-valued functions. Then the real roots of the equation $f(b) = 0$ are the right identity elements with respect to $*$.

Similarly, let $*$ be a binary operation on $\mathbb{R}$ expressible in the form

$$a * b = f(b)g(a) + b.$$

Then the real roots of the equation $g(a) = 0$ are the left identity elements with respect to $*$.

For example, the set of right identity elements of the operation $\star$ on $\mathbb{R}$ defined by $a \star b = a + a \sin b$ is $\{n\pi : n \text{ any integer}\}$ ; the set of left identity elements of the binary operation $\Delta$ on $\mathbb{R}$ defined by $a \Delta b = [a] + b$, where $[x]$ is the greatest integer not exceeding x, is $\{x \in \mathbb{R} : 0 \leqslant x < 1\}$. Note that in the latter, there are uncountably many left identity elements.

There are many interesting binary operations which can be defined on the following subsets of $\mathbb{R}$ :

$\mathbb{Z}$ , the set of integers.
$\mathbb{Z}_+$ , the set of positive integers.
$\mathbb{Q}$ , the set of rational numbers.
$\mathbb{Q}_+$ , the set of positive rational numbers.

For example, the binary operation $\Theta$ on $\mathbb{Z}_+$ defined by

$$a \Theta b = |1-a-b|$$

has 1 as a right and left identity element. But for the binary operation $\nabla$ on $\mathbb{Z}$ defined by

$$a \nabla b = |1-a-b|,$$

there exists no identity elements. (Consider negative elements of $\mathbb{Z}$.)

The above results can be extended to $\mathbb{C}$, the set of complex numbers.

Consider the following example.

Let $o$ be a binary operation on $\mathbb{C}$ defined by

$$a \circ b = (a^2 + 5)b.$$

This can be written as

$$a \circ b = (a^2 + 4)b + b$$
$$= (a - 2i)(a + 2i)b + b.$$

The left identity elements are 2i and -2i. There are no right identity elements.

We can easily extend the results to binary operations on matrices, such as on $M_{2\times2}(\mathbb{R})$, the set of 2 x 2 real matrices.

Let $*$ be a binary operation on $M_{2\times 2}(\mathbb{R})$ expressible in the form

$$A * B = A + g(A)f(B)$$

where f and g are functions from $M_{2\times 2}(\mathbb{R})$ to itself, and the operations on the right hand side are the ordinary matrix operations. Then the roots of the equation $f(B) = 0$ are the right identity elements with respect to $*$.

A similar remark is applicable to left identity elements.

As an illustration, consider the following: Let $\Delta$ be a binary operation on $M_{2\times 2}(\mathbb{R})$ defined by

$$A \Delta B = A(B^2 + 2B + 2I),$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Rewrite this in the form

$$A \Delta B = A + A(B + I)^2.$$

It is clear that the left identity element is $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. There are no right identity elements.

So far, we have considered only operations expressible in the form $a * b = a + g(a)f(b)$. This restriction is not necessary.

Consider $a \nabla b = a + b + \sin ab$ for $a, b \in \mathbb{R}$. This can be written as

$$a \nabla b = a + (b + \sin ab) \quad \text{or} \quad a \nabla b = (a + \sin ab) + b.$$

In either case, the expression in brackets cannot be written in the form $g(a)f(b)$. Nonetheless, using the same idea as before, by setting the expressions in brackets to zero identically, we obtain the right identity element $b = 0$ and the left identity element $a = 0$.

A similar remark applies in the case of matrices. Let $o$ be a binary operation on $M_{2\times 2}(\mathbb{R})$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} o \begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} a^2x + b^2u & a^2y + b^2v \\ c^2x + d^2u & c^2y + d^2v \end{pmatrix}$$

This is expressible in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} o \begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} (a^2 - 1)x + b^2u & (a^2 - 1)y + b^2v \\ c^2x + (d^2 - 1)u & c^2y + (d^2 - 1)v \end{pmatrix} + \begin{pmatrix} x & y \\ u & v \end{pmatrix}.$$

Without having to express the first matrix on the right hand side in the form $f(B)g(A)$, we see that there are four left identity elements given by $a = \pm 1$, $b = 0$, $c = 0$, $d = \pm 1$.

This idea can also be applied to binary operations on ordered pairs.

Let # be a binary operation on $\mathbb{R}^2$ defined by

$$(a,b) \# (c,d) = (ad, 3bc).$$

This is expressible in the form

$$(a,b) \# (c,d) = (a,b) + (a(d-1), b(3c-1)) \quad \ldots\ldots(1)$$

or

$$(a,b) \# (c,d) = (ad-c, 3bc-d) + (c,d) \quad \ldots\ldots(2)$$

From (1), we see that the right identity element is $(c,d) = (1/3, 1)$.
From (2), we see that, since there are no roots to the relations

$$ad-c = 0, \quad 3bc-d = 0 \text{ for all } (c,d) \in \mathbb{R}^2,$$

we have no left identity elements.

Finally, we mention the following simple results.

(i) Suppose $e_1$ and $e_2$ are left identity and right identity elements with respect to a binary operation $*$ defined on a non-empty set A. Then $e_1 = e_2$.

For, by definition, $e_1 * a = a$ for all $a \in A$ and $b * e_2 = b$ for all $b \in A$. Putting $a = e_2$ and $b = e_1$, we have $e_1 * e_2 = e_2$ and $e_1 * e_2 = e_1$ and hence $e_1 = e_2 = e$ (say).

The element e is known as the identity element with respect to $*$.

(ii) There exists no more than one identity element with respect to a given binary operation.

This is a consequence of (i).

As an example, the binary operation $\oplus$ on $\mathbb{R}$ defined by $a \oplus b = a + b - 1$ has 1 as the identity element since $a \oplus 1 = a$ and $1 \oplus b = b$ for all $a,b \in \mathbb{R}$.