WHAT IS A STAR?*

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For the best observed star, one can hope to get only four basic information, namely its mass, luminosity, composition of its outer layer and its radius. In addition, extensive and intensive astrophysical observations taken over the last three quarters of this century reveal that stars are unchanging constant configurations over a long interval of time. Though a period of 75 years in the life of a star is a rather minute span, geological observations on fossil algae point to the fact that the temperature of the earth about a billion years ago could not have differed from the present by more than 20^0C. Moreover, the study of the pulsations of Cepheid variables by highly accurate electronic devices also does not suggest any major change in the vast majority of these stars within a measurable time. So one may conclude that the interior of a star is more or less in perfect equilibrium, and the study of a star could in most cases be confined to that of equilibrium configurations only.

The description of a star will involve the equations of change describing the equilibrium state of a spherical star along with the appropriate number of constitutive relations. The equations of change consist of hydrostatic equilibrium equations, thermal equilibrium equation and energy transport equations.

A. Hydrostatic equilibrium

Consider a star as a spherical mass of gas. The hydrostatic equilibrium of any of its elementary volume is obtained by a balance between the outward pressure and the gravitational force directed inwards. Consider an elementary cylindrical volume at a distance r with cross-section ds and height dr (Fig. 1). Its hydrostatic equilibrium can be described by the equation

$$- \frac{dP}{dr} \, ds \, dr = G \frac{M_r}{r^2} \rho \, ds \, dr$$

i.e.

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2} \text{, ... (1)}$$

and

$$\frac{dM_r}{dr} = 4 \pi r^2 \rho \quad \text{..................(2)}$$

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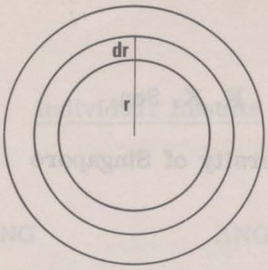

Fig. 1

The pressure P is a monotonic decreasing function of the distance r from the centre; M_r is the mass of the portion of the star interior to r; $\rho(r)$ is the density of the star, and G is the universal gravitational constant.

These two equations by themselves are not enough to give a unique distribution of pressure, density and mass in the interior of a star. However, they do allow us to get some insight into the orders of magnitude of pressure and temperature in the interior of a star.

Let us apply (1) to a point midway between the centre and the surface of our best known star, the Sun. At that point, we have

ρ : $\bar{\rho}_\odot$, mean density of the Sun,

M_r : $\frac{1}{2}M_\odot$, where $M_\odot$ is the solar mass,

r : $\frac{1}{2}R_\odot$, where $R_\odot$ is the solar radius,

dP : difference between the central pressure ($P_{c\odot}$) and the surface pressure ($P_{s\odot}$); the latter may be ignored in comparison with the former,

dr : radius of the Sun.

Note that dP/dr is linearly approximated by $P_{c\odot}/R_\odot$.

From (1), $P_{c\odot} = 2\bar{\rho}_\odot GM_\odot = 6 \times 10^{13}$ c.g.s. units ....(3)

From this estimate of pressure, one can obtain an estimate of the temperature, provided the equation of state is known. In this case, it is taken to be that of an ideal gas :

$$P = \frac{k}{m}\rho T \quad \text{..........................} \quad (4)$$

where k is the Boltzmann constant and m is the mean molecular weight of the gas. As hydrogen is the most abundant element in the main sequence stars like the Sun, m is that for hydrogen.

Now assuming the pressure at the midpoint is half that at the centre, the average temperature $\bar{T}_{\odot}$ is estimated to be of the order of $10^7\,{}^0K$. What does this rough estimate tell us? This tells us that :

(a) We have to work with gases and not solids or liquids.
(b) The gases are ionised.

Hydrostatic equilibrium alone does not ensure a constant star. In addition, there must be thermal equilibrium.

B. Thermal equilibrium

Perfect thermal equilibrium is attained in a system when all parts of it have reached the same temperature and flow of energy between parts does not occur. In a star, such perfect equilibrium is not possible. There is a great difference between surface and interior temperatures, and energy flows out of the surface of the star. So what kind of thermal equilibrium holds in the stars? To answer this question, we examine the energy stored inside which feeds the flux of energy through the star's surface. They consist of the thermal energy E_T and the gravitational energy E_G given by

$$E_T = \int_0^R (\frac{3}{2} kT/m)\, 4\pi r^2 \rho dr = (\frac{3}{2} \overline{kT/m})\, M, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$E_G = \int_0^R (-\,GM\,/r)\, 4\pi r^2 \rho dr = -\,\overline{(GM_r/r)} M \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The bar on the top indicates mean values. Using the rough figures of M, T, etc, for the Sun, one observes that E_T and E_G are of the same order of magnitude $\approx 10^{48}$ ergs.

We now follow a contracting star. Its gravitational energy decreases steadily. The net energy available for radiation from the surface is just equal to the thermal energy. The rate of radiative surface loss from the star is given by its luminosity L. Considering the estimated values of the store of thermal energy $E_{T\odot}$ and the radiative loss of energy from the surface $L_{\odot}$ for the Sun, one can predict the span of its life, which works out to be

$$E_{T\odot} / L_{\odot} \approx 10^{15} \text{ sec} \approx 3 \times 10^7 \text{ yrs.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

However, the geological estimate of the earth's age (which cannot possibly be less than that of the Sun) puts it at (at least) a billion years. So thermal energy alone cannot sustain the Sun for that many years. One must look for other sources of energy, In this, nuclear energy becomes handy. Most stars consist mainly of hydrogen which at the temperature of the stellar interior of $10^7\,{}^0K$ undergoes fusion to form helium :

$$4\,{}_1H^1 \longrightarrow {}_2He^4 + 2e^- + \text{energy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

The difference between the mass of reactant hydrogen and that of the product in this reaction is released as energy according to Einstein's mass-energy relation $E = mc^2$, where m is the mass defect. For the Sun, the estimated nuclear energy store $E_{N\odot}$ is of the order of 10^{52} ergs, and with this energy reserve, one can account for a life-span of the Sun of the order of

$$E_{N\odot} / L_{\odot} \approx 10^{11} \text{ yrs.} \quad \ldots\ldots (9)$$

Thus the store of nuclear energy is adequate to explain a billion years or more of the life span of stars like our Sun. The energy loss from the surface of a star (measured by its luminosity) is replenished by energy released by nuclear processes throughout the stellar interior. Thus we can write

$$L = \int_0^R \mathcal{E} \rho \, 4\pi r^2 dr \quad \ldots\ldots (10)$$

where $\mathcal{E}$ is the energy released by nuclear processes per unit area per unit time, and R is the radius of the star. However, one need not be worried about the continous fulfilment of the above condition. If for any reason, the nuclear energy source is turned off, the star will continue to shine, feeding on its store of gravitational energy. Equation (10) ensures the global energy balance of the star. However, the balance must hold for every section of the star. This can be stated as

$$\frac{dL_r}{dr} = \mathcal{E} \rho (4\pi r^2) \quad \ldots\ldots (11)$$

where L_r is the energy flux through the sphere of radius r. In some critical phases of stellar evolution, one has to take into account the variations of thermal and gravitational energy in addition to the nuclear energy. In such cases, equation (11) has to be replaced by a more general equation like

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \left[\mathcal{E} - \frac{3}{2} \rho^{2/3} \frac{d}{dt} (P \rho^{-5/3}) \right] \quad \ldots\ldots (12)$$

C. Energy transport condition

Thus far, our preoccupation was to obtain a balance between the energy flux and energy source. However, the energy flux is determined by the mechanism of energy transport such as conduction, convection and radiation. Essentially the existence of temperature gradient is the cause of energy transport. The main process of transfer of energy from the interior to the surface is radiative, convection playing a secondary role in certain classes of stars. Condition is too slow a process to play any accountable role.

If $I(r, \theta)$ is the intensity of radiation, being the energy per unit area per second per unit solid angle $d\omega$ of the radiation at a distance r from the centre, in a direction at an angle θ with the radius vector, the equation of radiative transfer can be written as

$$\frac{\partial I(r, \theta)}{\partial r} \cos\theta - \frac{\partial I(r, \theta)}{\partial \theta} \frac{\sin\theta}{r} = -\chi \rho I(r, \theta) + \frac{1}{4\pi} j\rho \quad \ldots\ldots (13)$$

where χ is the absorption coefficient per gm and j represents the total amount of energy emitted per gm per sec isotropically in all directions.

In stellar atmospheres, the solution of equation (13) is a major problem in itself. However, in the stellar interior, the radiation field is very nearly isotropic and the problem is greatly simplified. Instead of working with $I(r, \theta)$, one can, in this case, work with its three moments, namely

Energy density, $E(r) = c^{-1} \int I d\omega$ .... (14)

Radiation flux, $H(r) = \int I \cos\theta \, d\omega$ .... (15)

Radiation pressure, $P_R(r) = c^{-1} \int I \cos^2\theta \, d\omega$ .... (16)

where c is the velocity of light.

The first two moments of equation (13) can be written as

$$\frac{dH}{dr} + \frac{2}{r} H + c \chi \rho E - j\rho = 0 \qquad (17)$$

$$\frac{dP_R}{dr} + \frac{1}{r}(3P_R - E) + \frac{\chi\rho}{c} H = 0 \qquad (18)$$

We have in equations (17), (18) two equations in three unknowns E, H, P_R. This situation cannot be improved by taking another moment equation as this will introduce another unknown. What is needed is a relation between the moments, and this is achieved by representing $I(r, \theta)$ by the following series

$$I(r, \theta) = I_0 + I_1 \cos\theta + I_2 \cos^2\theta + \ldots, \qquad (19)$$

The convergence of the series is tested. Then truncating the series after the first two terms in the first approximation, one finds that

$$E = \frac{4\pi}{c} I_0 , \quad H = \frac{4\pi}{3} I_1 , \quad P_R = \frac{1}{3} E \qquad (20)$$

The relative errors of these relations is estimated to be of the order of 10^{-20}.

Now, remembering that $L_r = 4\pi r^2 H$, $dL_r/dr = \mathcal{E} 4\pi r^2$ and $j = \chi acT^4 + \mathcal{E}$ (a is the Stefan constant), one obtains after a series of eliminations

$$E = aT^4 \qquad (21)$$

$$P_R = \frac{1}{3} aT^4 \qquad (22)$$

And from (18), we get

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\chi\rho}{T^3} \frac{L_r}{4\pi r^2} \qquad (23)$$

This is the equation for radiative transport. For convective transport, the corresponding relation is given by

$$\frac{dT}{dr} = (1 - \frac{1}{\gamma}) \frac{T}{P} \frac{dP}{dr} \quad \text{...(24)}$$

where γ is the ratio of the specific heats.

D. Constitutive relations

In addition to the equations of change (1), (2), (11) and/or (12), (23) and/or (24), we have to take into account three constitutive relations which characterise specifically the behaviour of the gas. They are the equation of state, given by

$$P = P(\rho, T, \text{composition}), \quad \text{.................................... (25)}$$

measure of the absorption property, given by

$$\chi = \chi(\rho, T, \text{composition}), \quad \text{.................................... (26)}$$

and the equation of energy generation by nuclear processes

$$\varepsilon = \varepsilon(\rho, T, \text{composition}) \quad \text{.................................... (27)}$$

In most cases, the composition of the star is determined by the abundance of hydrogen and helium, which mainly constitute a star. However, more parameters can be introduced and may have to be introduced in the case of complex stars.

E. Boundary conditions

The equations of change and the constitutive relations must hold good at every layer of the star. However, we shall still have to add the boundary conditions for the solution of the problem. Assuming the star to be of radius R, we can state the following boundary conditions.

(a) It follows from the definition that at the centre of the star, characterised by $r = 0$,

$$M_r = 0 \text{ and } L_r = 0.$$

(b) At the surface, $r = R$,

$$T = 0 \text{ and } P = 0$$

in the radiative case, and

$$T = 0 \text{ and } P = kT^{5/2}$$

in the convective case, where $k = k(M, L, R, \text{composition})$.

F. Mathematical model of a star

Let us examine now if the equation of change consisting of hydrostatic equilibrium equations (1), (2), thermal equilibrium equations (11) or (13), transport equations (23) of (24), along with constitutive relations (25), (26), (27), and the boundary conditions E-(a), (b), give us a unique knowledge of the interior of a star. We have to express the four variables : pressure P, mass M, luminosity L, and temperature T as functions of r, the radial distance from the centre. We use the constitutive relations (25), (26), (27) to eliminate ρ, χ, ε from the equations of change. We get four non-linear first-order differential equations in four variables P, L, M, T. These together with the four boundary conditions given in E give us a typical, well-defined, tractable boundary value problem. If we remember that all the variables P, T, M, L and r are positive everywhere and if we rule out the unlikely possibility of mathematical degeneracy, we can conclude that the basic problem posed above has a unique solution. This is indeed a mathematician's answer to what a star is.

Reference

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