GROUP OBJECTS IN A CATEGORY WITH FINITE PRODUCTS*

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A group is commonly defined as a system consisting of a set together with a binary operation satisfying certain axioms. This definition of a group can also be presented by means of diagrams and arrows in the category of sets. With this 'diagrammatic' presentation, the notion of 'group objects' in a category with finite products was generated. Furthermore, since no explicit mention of the group elements is made in the definition of a group object, it can be applied to other circumstances. Thus, a 'group object' can represent a group, a topological group, or a Lie group depending on whether the category concerned is a category of sets, of topological spaces, or of differentiable manifolds. The main purpose of this note is to introduce the definition of a group object and to give a concrete example of such an object with no group elements involved, namely, a group object in the category of matrices.

A category $\mathcal{C}$ is said to have finite products if for any finite number of objects c_1, $c_2, \ldots, c_n$, the product diagram $c_1 \times c_2 \times \ldots \times c_n \xrightarrow{p_i} c_i$, $i = 1,2,\ldots,n$, exists in $\mathcal{C}$. In particular, if $\mathcal{C}$ has finite products, then $\mathcal{C}$ has a terminal object $\mathbb{1}$, i.e. the product of no objects in $\mathcal{C}$. If $\mathcal{C}$ is a category with finite products, then an object A in $\mathcal{C}$ is said to be a group object in $\mathcal{C}$ if A is equipped with the following arrows:

$$\mu : A \times A \rightarrow A \ ,$$

$$\eta : \mathbb{1} \rightarrow A \ ,$$

$$\nu : A \rightarrow A \ ,$$

such that the following diagrams are commutative:

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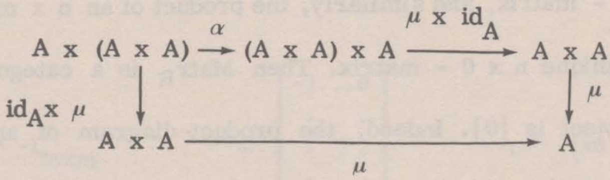
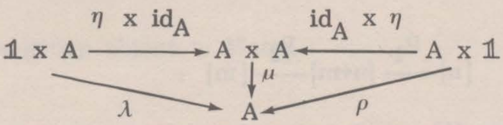


....(I)



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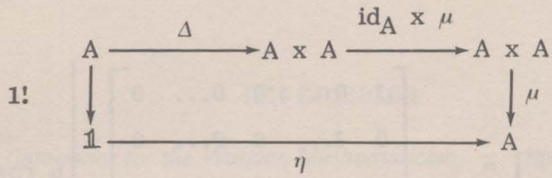
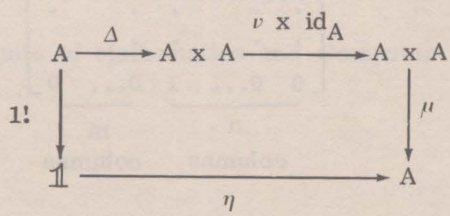


....(III a)

and



....(III b)

where α, λ and ρ are canonical isomorphisms in $\mathcal{C}$, id_A the identity arrow of A, Δ the diagonal arrow, and 1! denotes the unique arrow to the terminal object $\mathbb{1}$ from an object A in $\mathcal{C}$. To illustrate such a notion, we consider the category $Matr_R$ of matrices over the field of real numbers R. The objects of $Matr_R$ are non-negative integers [0], [1], [2], ... and for each pair of objects [n] and [m], the set of arrows Hom ([m], [n]) is the set of all n x m- matrices with entries from R. The composition of arrows is defined to be the usual multiplication of matrices.. Here, we define the set Hom ([0], [n]) *to be the* set consisting of the unique n x 0-matrix $[\ \ ]_{nx0}$, and similarly, Hom ([m], [0]) to be the set of the unique 0 x m - matrix $[\ ]_{0xm}$. Multiplication of such matrices are defined as follows. The product of the n x 0 - matrix with the 0 x m - matrix is the n x m - matrix with only zero's as its entries, whereas the product of the 0 x n - matrix and an n x m -

matrix is the unique $0 \times m$ - matrix, and similarly, the product of an $n \times m$ - matrix and the $m \times 0$ - matrix is the unique $n \times 0$ - matrix. Then $Matr_R$ is a category with finite products and its terminal object is $[0]$. Indeed, the product-diagram of any two objects $[n]$ and $[m]$ in $Matr_R$ is given by

$$[n] \xleftarrow{p_1} [n+m] \xrightarrow{p_2} [m] \quad :$$

where p_1 and p_2 are the respective $n \times (n+m)$ and $m \times (n+m)$ matrices defined as follows:

$$p_1 = \left[I_{nxn} \quad 0_{nxm} \right] = \underbrace{\begin{bmatrix} 1 & 0 \ldots & 0 & 0 \ldots & 0 \\ 0 & 1 \ldots & 0 & 0 \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 \ldots & 1 & 0 \ldots & 0 \end{bmatrix}}_{\text{n columns} \quad \text{m columns}} \Bigg\} \text{ n rows}$$

and

$$p_2 = \left[0_{mxn} \quad I_{mxm} \right] = \underbrace{\begin{bmatrix} 0 \ldots 0 & 1 & 0 \ldots 0 \\ 0 \ldots 0 & 0 & 1 \ldots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 \ldots 0 & 0 & 0 \ldots 1 \end{bmatrix}}_{\text{n columns} \quad \text{m columns}} \Bigg\} \text{ m rows}$$

One can readily verify that for any object $[m]$, $m \neq 0$, in $Matr_R$, the following arrows:

$$\mu = \left[I_{mxm} \; I_{mxm} \right] = \begin{bmatrix} 1 \ldots 0 & 1 \ldots 0 \\ \cdot \quad \cdot & \cdot \quad \cdot \\ \cdot \quad \cdot & \cdot \quad \cdot \\ 0 \ldots 1 & 0 \ldots 1 \end{bmatrix} \quad : \quad [m] \times [m] \longrightarrow [m]$$

$$\eta = [\quad]_{mx0} \quad : \quad [0] \longrightarrow [m]$$

and

$$\nu = -I_{mxm} = \begin{bmatrix} -1 \ldots 0 \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ 0 \quad -1 \end{bmatrix} : [m] \to [m]$$

will make [m] into a group object in $Matr_R$.

REFERENCES

1. S. MacLane, *Categories for the Working Mathematician,* Springer-Verlag GTM 5, 1971.

2. S.K. Tan, *"Abelian Group Objects in a Category of Matrices",* Research Report No. 52, Lee Kong Chian Institute of Mathematics and Computer Science, Nanyang University, October, 1977.