ON A PROBLEM OF IAN D. MACDONALD*

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In a recent paper, "Factor groups", published in Mathematical Gazette 62 (1978), 29 - 35, Ian D. Macdonald of the University of Stirling, Scotland, gives a new approach of introducing the concept of "normal subgroups", which often presents difficulties to the beginning student.

I use the notation $\mathcal{G}$ for a group (endowed with a structure) and the notation $G$ for the set of elements of the group $G$ (devoid of structure). $G$ is then called the "carrier" of $\mathcal{G}$. If $H$ is a subset of $G$ and $x \in G$, the set

$$Hx = \{ hx \mid h \in H \}$$

is called a (right) coset of $H$. Macdonald introduces two types of multiplication for cosets.

The first type of multiplication is given by

$$Hx \cdot Hy = Hxy.$$

What sort of algebraic system does this give? It turns out that this multiplication of cosets is a group multiplication.

EXAMPLE. Let $\mathcal{G} = S_3$, the symmetric group on 3 symbols, and $H = \{(1,2,3), (1,3,2)\}$. There are 6 right cosets of $H$:

$$H, \ \{(1,1,3),(2,2,3)\}, \ \{(1,2,1),(3,2,3)\}, \ \{(1,2),(1,3)\}. $$

When \( n = 2 \), we have
\[ HxHyH = Hxy. \]

For \( x = y = e \), we have
\[ H^3 = H. \]

The question is whether or not \( H^3 = H \) implies that \( H^2 = H \). The answer is "No". Take \( H = \{h\} \) where \( h^2 = e, h \neq e \).

Next, we can rewrite (1') in the form.

\[
H^{-1} H^2 \cdots H^{-n} H^{2n} = H^{n+1}.
\]

The second question asked by Isaacs is the following.

**QUESTION.** Assume that to all \( x, y \in G \), there is \( z \in G \) such that \( HxHy = Hz \). Must \( H \) be "normal" in \( G \)?

The answer is again "No".

**COUNTER-EXAMPLE.**

Let \( G = \left\{ \begin{pmatrix} r & 0 \\ s & 1 \end{pmatrix} \mid r, s \in \mathbb{Q}, r > 0 \right\} \), and \( \not\) the group with carrier \( G \) under matrix multiplication.

Consider the (singleton) set
\[ H = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad a \neq 0. \]
Denote the element of $H$ by $h$. If $x = \begin{pmatrix} r & 0 \\ s & 1 \end{pmatrix}$, then

$$x^{-1} = \begin{pmatrix} r^{-1} & 0 \\ -sr^{-1} & 1 \end{pmatrix}.$$ We have

$$h^x = \begin{pmatrix} r^{-1} & 0 \\ -sr^{-1} + a & 1 \end{pmatrix} \begin{pmatrix} r & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ra & 1 \end{pmatrix}$$

If $y = \begin{pmatrix} t & 0 \\ 1 & 1 \end{pmatrix}$, we have

$$h^xh^y = \begin{pmatrix} 1 & 0 \\ ra & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (r+t)a & 1 \end{pmatrix} = h^z,$$

where $z = \begin{pmatrix} r+t & 0 \\ * & 1 \end{pmatrix}$.

However, $H$ is clearly not "normal" in $G$.

In the above counter-example, $H$ has only one element. Is it possible to obtain a counter-example in which $H$ contains more than one element? Indeed we can. We can always choose $H$ to be a set of the form.

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \right\} \rho \prec a \prec \sigma$$

where $\rho, \sigma$ are real numbers and may be $+\infty$ or $-\infty$. One or more of the strict inequalities which define $H$ may be replaced by $\leq$. It is then clear that there are $\aleph_0$ counter-examples in $\mathcal{G}$. 