ON A PROBLEM OF IAN D. MACDONALD*

B. H. Neumann

Australian National University, Canberra, and
Division of Mathematics and Statistics, CSIRO, Canberra

In a recent paper, "Factor groups", published in Mathematical Gazette 62 (1978), 29 - 35, Ian D. Macdonald of the University of Stirling, Scotland, gives a new approach of introducing the concept of "normal subgroups", which often presents difficulties to the beginning student.

I use the notation $\mathscr{G}$ for a group (endowed with a structure) and the notation G for the set of elements of the group G (devoid of structure). G is then called the "carrier" of $\mathscr{G}$. If H is a subset of G and $x \in G$, the set

$$Hx = \left\{ hx \;\middle|\; h \in H \right\}$$

is called a (right) coset of H. Macdonald introduces two types of multiplication for cosets.

The first type of multiplication is given by

$$Hx \,.\, Hy = Hxy.$$

What sort of algebraic system does this give? It turns out that this multiplication of cosets is a group multiplication.

EXAMPLE. Let $\mathscr{G} = S_3$, the symmetric group on 3 symbols, and $H = \left\{(1,2,3),\ (1,3,2)\right\}$. There are 6 right cosets of H:

$$H,\ \left\{1,(1,3,2)\right\},\ \left\{1,(1,2,3)\right\},$$

$$\left\{(1,3),\ (2,3)\right\},\ \left\{(1,2),\ (2,3)\right\},\ \left\{(1,2),\ (1,3)\right\}.$$

---

* Invited lecture delivered on 2 September 1978 at the Seminar on Group Theory and Related Topics. Notes taken by Y. K. Leong.

When n = 2, we have

$$HxHyH = Hxy.$$

For x = y = e, we have

$$H^3 = H.$$

The question is whether or not $H^3 = H$ implies that $H^2 = H$. The answer is "No". Take $H = \{h\}$ where $h^2 = e$, $h \neq e$.

Next, we can rewrite (1') in the form.

$$H^{y_1}H^{y_2}\ldots H^{y_n}H^{y_{n+1}} = H^{y_{n+1}}. \tag{2'}$$

The second question asked by Isaacs is the following.

QUESTION. Assume that to all $x, y \in G$, there is $z \in G$ such that $H^x H^y = H^z$. Must H be "normal" in G?

The answer is again "No".

COUNTER-EXAMPLE.

Let

$$G = \left\{ \begin{pmatrix} r & 0 \\ s & 1 \end{pmatrix} \;\middle|\; r, s \in Q; r > 0 \right\},$$

and $\mathscr{G}$ the group with carrier G under matrix multiplication.

Consider the (singleton) set

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \right\}, \ a \neq 0.$$

Denote the element of H by h. If $x = \begin{pmatrix} r & 0 \\ s & 1 \end{pmatrix}$, then

$x^{-1} = \begin{pmatrix} r^{-1} & 0 \\ -sr^{-1} & 1 \end{pmatrix}$. We have

$$h^x = \begin{pmatrix} r^{-1} & 0 \\ -sr^{-1}+a & 1 \end{pmatrix} \begin{pmatrix} r & 0 \\ s & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ ra & 1 \end{pmatrix}$$

If $y = \begin{pmatrix} t & 0 \\ x & 1 \end{pmatrix}$, we have

$$h^x h^y = \begin{pmatrix} 1 & 0 \\ ra & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ (r+t)a & 1 \end{pmatrix}$$

$$= h^z,$$

where $z = \begin{pmatrix} r+t & 0 \\ * & 1 \end{pmatrix}$.

However, H is clearly not "normal" in G.

In the above counter-example, H has only one element. Is it possible to obtain a counter-example in which H contains more than one element? Indeed we can. We can always choose H to be a set of the form.

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \;\middle|\; \rho < a < \sigma \right\}$$

where ρ, σ are real numbers and may be $+\infty$ or $-\infty$. One or more of the strict inequalities which define H may be replaced by $\leqslant$. It is then clear that there are $2^{\aleph_0}$ counter-examples in $\mathcal{G}$.