SOME NEW RUMOURS IN GROUP THEORY*

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In this talk, I shall help to spread some group theory rumours picked up at the recent International Congress of Mathematicians at Helsinki. My notebook containing the details of these results was unfortunately lost at the Congress and has not yet been recovered, so I shall speak entirely from memory.

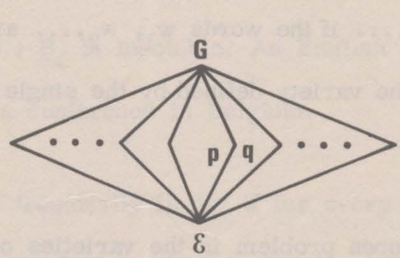
There is a long-standing problem attributed to the logician Alfred Tarski:

"Is there an infinite group with subgroup lattice of height 2?"

Equivalently:

"Is there an infinite group all of whose proper subgroups are of order p for some prime p?"

Such a group, if it exists, is often called a "Tarski Monster". Its lattice of subgroups would look like



where p, q, ... are primes which may or may not be distinct.

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Recently, the Israeli mathematician I. A. Rips has discovered such a "Tarski Monster" with a single prime p at the bottom (of the subgroup lattice) for some $p > 10^{10,000}$. He uses methods in "small cancellation theory" which was started some thirty years ago and has since been developed by various people, notably J.L. Britton, R.C. Lyndon and P.E. Schupp. Rips has described his proof to Schupp who has checked the details and seems to have been convinced.

A group G may be considered as a set of words in the letters a, b,..., subjected to certain relations R, S,... between a, b,... We may then write G in the form of a presentation:

$$G = \left\langle a, b, \ldots \; ; \; R, S, \ldots \right\rangle .$$

Let w be a word in the symbols a, b,... . If w reduces to the identity element in G whenever elements of G are substituted for the symbols a, b,... we say that w is a law or identical relation in G.

A collection of (isomorphism classes of) groups is said to be a variety of groups defined by the laws w_1, w_2,... if the words w_1, w_2,... are all laws in each group of this collection. For instance, the variety defined by the single word $a^{-1} b^{-1} ab$ is the variety of abelian groups.

The following is an open problem in the varieties of groups:

"Is there a variety of groups, NOT the variety of abelian groups, in which all finite groups are abelian?"

Rips has claimed to have made some headway towards its solution. The following result has been announced, although Rips himself considers his proof to be incomplete.

(Rips) Let G_1, G_2, ... be a countably infinite set of non-abelian groups. Then there is a non-abelian variety of groups that contains none of G_1, G_2, ...

Groups of exponent n form a variety defined by the word x^n, called the Burnside variety of exponent n and denoted by $\underline{\underline{B}}_n$. It is known that the order of the free d-generator group in $\underline{\underline{B}}_2$ is 2^d, while a result of F.W. Levi and B.L. van der Waerden tells us that the order of the free d-generator group in $\underline{\underline{B}}_3$ is $3^{\binom{d}{1}+\binom{d}{2}+\binom{d}{3}}$. Around 1940, I.N. Sanov proved that finitely generated groups in $\underline{\underline{B}}_4$ are finite.

M.F. Newman and G. Havas have reconfirmed a result of W. Burnside and S. Tobin that the order of the free 2-generator group in $\underline{\underline{B}}_4$ is 2^{12}. They have computed with the aid of a computer the order of the largest group on 4 generators in $\underline{\underline{B}}_4$, which is found to be 2^{422}. If the order had turned out to be smaller, one could have derived from a result of N.D. Gupta that $\underline{\underline{B}}_4$ is soluble.

The question : "Is $\underline{\underline{B}}_4$ soluble?" has been answered by a Russian mathematician Razmyslov. The answer is NO : $\underline{\underline{B}}_4$ is insoluble. An English summary of the proof has been given by G. Bergman in a conference in Belgium.

A group G is said to be 'residually finite' if for every g, $h \in G$, $g \neq h$, there is an epimorphism $\phi : G \longrightarrow G^*$ with G^* finite and such that $\phi(g) \neq \phi(h)$.

N.D. Gupta posed the following question:

"Let G be an extension of a free product of cyclic groups by a cyclic group. Is G residually finite?"

I have proved (unpublished) that an extension of a free product of finite groups by a cyclic group is residually finite.

That the answer to the general question is again in the affirmative under certain conditions is due to B. Baumslag, Narain Gupta, and F. Levin, and their proof will appear in the Bulletin of the Australian Mathematical Society, Vol. 19 (1978).