1. **About Daniel Pedoe (adapted from [1])**

Daniel Pedoe is Professor of Mathematics at the University of Minnesota. Born and educated in England, he received his Ph.D. from Cambridge University in 1937, having spent a year as a Member of the Institute for Advanced Study, Princeton. In 1947 he was awarded a Leverhulme Research Fellowship, and returned to Cambridge to work with Sir William Hodge on ‘Methods of Algebraic Geometry’ (Cambridge University Press). The three volumes of this highly regarded text have been translated into Russian, and the first two volumes have been reissued by the Cambridge University Press in paperback. Professor Pedoe has held professorships in Sudan and in Singapore, and became resident in the United States in 1962. He is the author of several mathematics books, all of which show his deep interest in geometry. That he has a gift for exposition is shown by the success of ‘The Gentle Art of Mathematics’, published by Penguin Books, and the award of a Lester R. Ford prize for exposition by the Mathematical Association of America.

2. **Motivation**

The Pedoe’s theorem which we are going to discuss gives an inequality relating the six sides of two triangles and their areas. More precisely, on the greater side we have a symmetric expression relating the six sides of two triangles, and on the smaller side we have a symmetric expression relating their areas. Our approach of proving this Pedoe’s theorem is as follows: we first make use of Cauchy’s Inequality to transform the greater side to an intermediate expression of smaller magnitude, which is also symmetric in terms of the six sides of two triangles but free from interaction between one another, and then prove that this intermediate expression is still greater than or equal to the smaller side in Pedoe’s inequality. Moreover, the respective necessary and sufficient condition for each of the equality signs to hold has also been obtained. This intermediate expression has thus become an improved bound of Pedoe’s Theorem.

3. **Notation and basic lemmas**

Throughout this note, \( \Delta ABC \) and \( \Delta A'B'C' \) denote two arbitrary triangles. As usual, \( a, b, \) and \( c \) are the three sides of \( \Delta ABC \) opposite to the three interior angles \( A, B, \) and \( C \) respectively. Let the area of \( \Delta ABC \) be denoted by \( \Delta \). Likewise, \( a', b', \) \( c', A', B', \) \( C' \), and \( \Delta' \) are defined for \( \Delta A'B'C' \).

Next, \( \Sigma a^2 \) denotes the sum which is a symmetric expression in which \( a^2 \) is a representative term, that is, \( \Sigma a^2 = a^2 + b^2 + c^2 \). Likewise, we have \( \Sigma (a^2 a'^2) = a^2 a'^2 + b^2 b'^2 + c^2 c'^2 \), \( \Sigma \cot A = \cot A + \cot B + \cot C \), etc.
Throughout this note, the expressions $a'^2 (b^2 + c^2 - a^2) + b'^2 (c^2 + a^2 - b^2) + c'^2 (a^2 + b^2 - c^2)$ and $(\Sigma a^2)(\Sigma a'^2) - 2\sqrt{\Sigma a^4}(\Sigma a'^4)$ are denoted by $D$ and $E$ respectively.

Now, we state and prove a special case of Cauchy's inequality. The following proof can be understood by anyone who knows about the scalar product of vectors at G.C.E. 'A' Level.

Lemma 1. $\Sigma (a^2 a'^2) \leq \sqrt{(\Sigma a^4)(\Sigma a'^4)}$ and the equality holds if and only if $\Delta ABC \sim \Delta A'B'C'$.

Proof. Consider the two vectors $u = (a^2, b^2, c^2)$ and $v = (a'^2, b^2, c'^2)$ in $\mathbb{R}^3$. We have $u \cdot v \leq |u||v|$ and the equality holds if and only if $u = \lambda v$ for some positive constant $\lambda$, hence the result.

Lemma 2. $D \geq E$ and equality holds if and only if $\Delta ABC \sim \Delta A'B'C'$.

Proof. $D = a'^2 (\Sigma a^2 - 2a'^2) + b'^2 (\Sigma a^2 - 2b^2) + c'^2 (\Sigma a^2 - 2c^2)$

$= (\Sigma a^2)(\Sigma a'^2) - 2\Sigma (a^2 a'^2)$

$\geq (\Sigma a^2)(\Sigma a'^2) - 2\sqrt{\Sigma a^4}(\Sigma a'^4)$ (by Lemma 1)

$= E$.

Clearly, $D = E$ if and only if $\Delta ABC \sim \Delta A'B'C'$ by Lemma 1.

Lemma 3. $\Sigma \tan A = \tan A \tan B \tan C$.

Proof. $\Sigma \tan A = \tan A + \tan B + \tan C$

$= \tan A + \tan B - \tan (A + B)$ (as $C = 180^\circ - (A + B)$)

$= \tan A + \tan B - \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= (\tan A + \tan B)(1 - \frac{1}{1 - \tan A \tan B})$

$= (\tan A + \tan B) \cdot (\frac{-\tan A \tan B}{1 - \tan A \tan B})$

$= \tan A \tan B \cdot (\frac{-\tan A + \tan B}{1 - \tan A \tan B})$

$= \tan A \tan B \tan C$. 

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Lemma 4. $\Sigma (\cot A \cot B) = 1.$

Proof. $\Sigma (\cot A \cot B) = \Sigma \left( \frac{1}{\tan A \tan B} \right)$

$= \frac{\Sigma \tan A}{\tan A \tan B \tan C}$ by Lemma 3.

Lemma 5. $\Sigma a^2 = 4 \Delta \Sigma \cot A.$

Proof. $\Sigma a^2 = (a^2 + b^2 - c^2) + (b^2 + c^2 - a^2) + (c^2 + a^2 - b^2)$

$= 2ab \cos C + 2bc \cos A + 2ac \cos B$ (by Cosine Rule)

$= 4 \cdot \frac{1}{2} ab \sin C \cot C + 4 \cdot \frac{1}{2} bc \sin A \cot A + 4 \cdot \frac{1}{2} ac \sin B \cot B$

$= 4 \Delta (\cot C + \cot A + \cot B)$ (as $\Delta = \frac{1}{2} ab \sin C$, etc.)

$= 4 \Delta \Sigma \cot A.$

Lemma 6. $\Sigma a^4 = 8\Delta^2 [ (\Sigma \cot A)^2 - 1].$

Proof. $\Sigma a^4 = (\Sigma a^2)^2 - 2\Sigma (a^2 b^2),$

$ab = 2 \cdot \frac{1}{2} ab \sin C \csc C = 2\Delta \csc C.$

Thus $\Sigma (a^2 b^2) = 4\Delta^2 \Sigma \csc^2 A$

$= 4\Delta^2 \Sigma (1 + \cot^2 A)$

$= 4\Delta^2 [3 + (\Sigma \cot A)^2 - 2\Sigma (\cot A \cot B)]$

$= 4\Delta^2 [1 + (\Sigma \cot A)^2]$ by Lemma 4.

Hence $\Sigma a^4 = 16\Delta^2 (\Sigma \cot A)^2 - 8\Delta^2 [1 + (\Sigma \cot A)^2]$ (by Lemma 5 and above)

$= 8\Delta^2 [ (\Sigma \cot A)^2 - 1].$

Lemma 7. $\Sigma \cot A \geq \sqrt{3}.$

Proof. If $A = B = C = 60^\circ$, the result is clearly true. Otherwise there exists at least one angle greater than $60^\circ$. Without loss of generality, let $B > 60^\circ$. Construct an equilateral triangle $A'B'C$ as shown in Figure 1.

Considering $\Delta AA'B$, we have

$$AA'^2 = a^2 + c^2 - 2ac \cos (B - 60^\circ)$$

$$= a^2 + c^2 - 2ac \left( \cos B \cdot \frac{1}{2} + \sin B \cdot \frac{\sqrt{3}}{2} \right)$$
\[ \begin{align*}
&= a^2 + c^2 - \frac{1}{2}(a^2 + c^2 - b^2) - 2\sqrt{3} \cdot \frac{1}{2} \sin B \\
&= \frac{1}{2}(a^2 + b^2 + c^2) - 2\sqrt{3} \Delta (b^2 - \Delta - \frac{1}{2}(a^2 + b^2 - c^2)) \\
&= 2 \Delta \Sigma \cot A - 2\sqrt{3} \Delta \quad \text{ (by Lemma 5)} \\
&= 2 \Delta (\Sigma \cot A - \sqrt{3}).
\end{align*} \]

Since \( AA' > 0 \), we must have \( \Sigma \cot A > \sqrt{3} \). The proof is complete.

**Fig. 1**

4. Intermediate expression for Pedoe’s Inequality

The original form of Pedoe’s Theorem states that for any two triangles \( \triangle ABC \) and \( \triangle A'B'C' \), \( D \geq 16\Delta \Delta' \) and the equality holds if and only if \( \triangle ABC \sim \triangle A'B'C' \). We now provide an intermediate expression between \( D \) and \( 16\Delta \Delta' \).

**Extended Pedoe’s Theorem**

Let \( \triangle ABC \) and \( \triangle A'B'C' \) be any two triangles. Let \( A \) and \( C' \) denote the areas of \( \triangle ABC \) and \( \triangle A'B'C' \) respectively.

Let \( D = a^2 \left( b^2 + c^2 - a^2 \right) + b^2 \left( c^2 + a^2 - b^2 \right) + c^2 \left( a^2 + b^2 - c^2 \right) \)
and \( E = \left( \Sigma a^2 \right) \left( \Sigma a'^2 \right) - 2\sqrt{\left( \Sigma a^2 \right) \left( \Sigma a'^2 \right)}. \)

Then \( D \geq E \geq 16\Delta \Delta' \), and \( E = 16\Delta \Delta' \) if and only if \( \Sigma \cot A = \Sigma \cot A' \). Moreover, the following are equivalent:

(1) \( D = 16\Delta \Delta' \)
(2) \( D = E \)
(3) \( \triangle ABC \sim \triangle A'B'C' \).

**Proof.** By Lemma 2, \( D \geq E \). Next, by Lemmas 5 and 6, we have

\[ \begin{align*}
E &= 16\Delta \Delta' (\Sigma \cot A)(\Sigma \cot A') - 16\Delta \Delta' \sqrt{[\left( \Sigma \cot A \right)^2 - 1]} \left[ (\Sigma \cot A')^2 - 1 \right] \\
&= 16\Delta \Delta' (\Sigma \cot A)(\Sigma \cot A') - \sqrt{[\left( \Sigma \cot A \right)^2 - 1]} \left[ (\Sigma \cot A')^2 - 1 \right].
\end{align*} \]
Since 
\[ (\Sigma \cot A)(\Sigma \cot A') - 1 \] ^2 = (\Sigma \cot A - \Sigma \cot A')^2 \geq 0,
we have 
\[ (\Sigma \cot A)(\Sigma \cot A') - 1 \] ^2 \geq (\Sigma \cot A)^2 - 1. 
By Lemma 7, \( (\Sigma \cot A)(\Sigma \cot A') - 1 \) \geq \( 3 - 1 \) \( 3 - 1 \) = 4 > 0.

Therefore \( (\Sigma \cot A)(\Sigma \cot A') - 1 \) \geq \( \sqrt{3 \cdot \sqrt{3} - 1} \) = 2 > 0.

Hence \( (\Sigma \cot A)(\Sigma \cot A') - \sqrt{3} \) \( \sqrt{3} - 1 \) \( 3 - 1 \) \( 3 - 1 \) = 4 > 0.

Remark. As illustrated in the following example, the difference between \( D \) and \( \Sigma \) can be quite large when compared with that between \( E \) and \( \Sigma \). For \( \triangle ABC \) take \( a = 3, b = 4, c = 5 \), and for \( \triangle A'B'C' \), take \( a' = 6, b' = 7, c' = 8 \). Then
\[ D = 2034 > E = 1974 > \Sigma = 1952. \]

References
