PROBLEMS OF INTERSCHOOL MATHEMATICAL COMPETITION 1983

PART A

Saturday, 2 July 1983 0930 – 1030

Attempt as many questions as you can. Circle your answers on the Answer Sheet provided.

Each question carries 5 marks.

1. The greatest common divisor of 878787878787 and 787878787878 is
   (a) 1,
   (b) 3,
   (c) 9,
   (d) 10101010101,
   (e) none of the above.

2. A is a 30-digit positive integer and B is a 20-digit positive integer. If the digits of A and B are all 9's, how many digits of the product AB are 8's?
   (a) none,
   (b) 1,
   (c) 60,
   (d) 300,
   (e) none of the above.

3. If "a|b" means "a divides b", where a and b are integers, determine which of the following statements is **false**.
   (a) $6|(a^3 - a)$,
   (b) If $a|(b^2 - 1)$, then $a|(b^4 - 1)$.
   (c) If $a^3|b^3$, then $a|b$.
   (d) If $a^2|b^3$, then $a|b$.
   (e) If $a^3|b^2$, then $a|b$

4. The decimal representation of 1000! (= 1.2.3 . . . 1000) ends in
   (a) 111 zeros,
   (b) 200 zeros,
   (c) 240 zeros,
   (d) 249 zeros,
   (e) 256 zeros.

5. A positive integer N is written at random. What is the probability that N is relatively prime to 12 (that is, N and 12 have no positive divisors other than 1)?

(a) $\frac{1}{2}$,

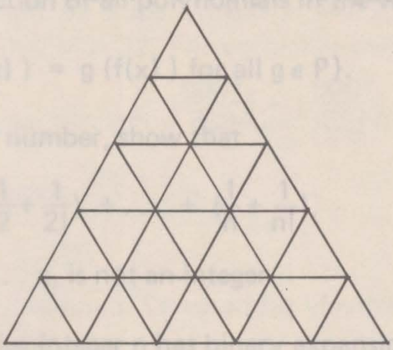
(b) $\frac{1}{6}$,

(c) $\frac{1}{12}$,

(d) $\frac{1}{3}$,

(e) $\frac{2}{3}$.

6. Find the number of triangles in the following diagram:



(a) 46,
(b) 48,
(c) 49,
(d) 51,
(e) 52.

7. Singapore, Malaysia, Thailand and Indonesia take part in a football competition in which each pair of teams plays one match. The winning team of a match is awarded 2 points while the losing team gets no point. If a match ends in a draw, each team gets 1 point.

Joe turns on the radio just as the announcer finishes reading the results: ". . . and Indonesia came fourth. So no two teams obtained the same number of points and the only drawn game was that of Malaysia versus Thailand".

From the information given, Joe was able to deduce that Singapore's total number of points and placing are:

(a) 6 points, 1st place,
(b) 4 points, 1st place,
(c) 4 points, 2nd place,
(d) 2 points, 2nd place,
(e) 2 points, 3rd place.

8. A polynomial f is such that

$$\int_0^1 f(x)dx = \int_0^1 x\, f(x)dx = \int_0^1 x^2 f(x)dx = 0,$$

$$\int_0^1 x^3 f(x)dx = 1.$$

Then the maximum value M of $|f(x)|$ in the interval $0 \leqslant x \leqslant 1$ satisfies

(a) $M < 4$,
(b) $M \leqslant 4 < 8$,
(c) $8 \leqslant M < 16$,
(d) $16 \leqslant M < 32$,
(e) $M \geqslant 32$.

9. The two hands of a clock coincide. When they coincide the next time, the hour hand will have moved through an angle of θ°, where

(a) $\theta > 32\frac{1}{2}$,

(b) $32 < \theta \leqslant 32\frac{1}{2}$,

(c) $31\frac{1}{2} < \theta \leqslant 32$,

(d) $31 < \theta \leqslant 31\frac{1}{2}$,

(e) $\theta \leqslant 31$.

10. Three socks are drawn at random from a cabinet containing an equal number of white, red and black socks. If p is the probability that at least 2 of the socks drawn are of the same colour, then the approximate value of p when the number of socks is large is

(a) $\frac{1}{2}$,

(b) $\frac{7}{12}$,

(c) $\frac{2}{3}$,

(d) $\frac{7}{9}$,

(e) $\frac{5}{6}$.

PART B

Saturday, 2 July 1983 1030 – 1230

Attempt as many questions as you can.

Each question carries 25 marks.

1. Let m be an integer greater than 1 and define the numbers $m_1, m_2, m_3, \ldots$ as follows:

$$m_1 = m, \quad m_{i+1} = m_i^2 - m_i + 1, \; i = 1, 2, \ldots$$

Show that none of the numbers $m_2, m_3, m_4, \ldots$ is divisible by m.

2. Let P be the collection of all polynomials in the variable x. Find the set

$$\{f \in P \mid f(g(x)) = g(f(x)) \text{ for all } g \in P\}.$$

3. If n is not a prime number, show that

$$(1 + \frac{1}{1!}) + (\frac{1}{2} + \frac{1}{2!}) + \ldots + (\frac{1}{n} + \frac{1}{n!}),$$

where $x! = 1.2.3 \ldots x$, is not an integer.

4. Suppose the positive integer n has binary expansion

$$n = a_m 2^m + a_{m-1} 2^{m-1} + \ldots + a_0.$$

Show that

$$[\frac{n}{2^k} + \frac{1}{2}] = 0$$

whenever $k \geqslant n+2$. Show further that

$$n = [\frac{n}{2} + \frac{1}{2}] + [\frac{n}{2^2} + \frac{1}{2}] + \ldots + [\frac{n}{2^n} + \frac{1}{2}].$$

(Here [x] denotes the greatest integer less than or equal to x.)

5. Let $E_1 = 2$ and $E_n = 2^{E_{n-1}}$, $n = 2, 3, \ldots$, and let $X_n = E_n - E_{n-1}$, $n = 2, 3, \ldots$ Prove that x_{n-1} divides X_n for $n \geqslant 3$.

6. Given n points $A_1, \ldots, A_n$ on a plane such that all the distances between pairs of points A_i, A_j, $i \neq j$, are different. For each $i = 1, 2, \ldots, n$, a line is drawn from A_i to its nearest neighbour. Show that no closed polygon is formed in the process and that each point A_i is joined to at most 5 other points.