SOLVE A DIFFERENTIAL EQUATION BY DIFFERENTIATION

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Under usual situation integration operations cannot be avoided when one wants to solve a nonhomogeneous differential equation. This is due to the fact that an integration operation is in nature a reverse operation of differentiation. Nevertheless, given a linear differential equation of the form

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_i \frac{d^{n-i} y}{dt^{n-i}} + \ldots + a_n y = e^{bt},$$

where a_0 $(\neq 0)$, $a_1, a_2, \ldots, a_n$, and b are complex or real constants, it is interesting that we have a chance to find a particular integral of the differential equation by differentiating the characteristic polynomial of the equation. We shall first establish the following theorem which is quite general and has never been seen in any textbook or paper. At the end, we also supply some typical examples which are instructive for undergraduate students.

THEOREM. Let $D \equiv \frac{d}{dt}$ *and the linear differential equation be*

$L(D)y(t) \equiv (a_0 D^n + a_1 D^{n-1} + \ldots + a_i D^{n-i} + \ldots + a_n) y(t) = e^{bt}$, $a_0 (\neq 0), a_1, a_2, \ldots, a_n$ *and* b *being complex or real constants. Then a particular integral* $y_p(t)$ *of the differential equation* $L(D)y(t) = e^{bt}$ *is*

$$\frac{t^m e^{bt}}{L^{(m)}(b)}$$

where $m = 0$ *if* $L(b) \neq 0$, *otherwise differentiating the characteristic polynomial* $L(s)$ *of the differential equation until reaching a positive integer* m *such that*

$$L^{(m)}(b) \equiv \left(\frac{d^m L(s)}{ds^m}\right)_{s=b} \neq 0$$

is first time satisfied.

Proof. Using D-operator method, we have

$$y_p(t) = \frac{1}{L(D)} e^{bt} = e^{bt} \frac{1}{L(D+b)} 1$$

$$= \frac{e^{bt}}{L(b)}, \text{ if } L(b) \neq 0.$$

This is the case for m = 0.

If $L(b) = 0$, we consider $L(D) = (D-b)^m G(D)$, where $G(D)$ is a polynomial in D with $G(b) \neq 0$, and m is a positive integer. Then we have

$$y_p(t) = \frac{1}{L(D)} e^{bt} = \frac{1}{(D-b)^m G(D)} e^{bt}$$

$$= \frac{1}{(D-b)^m} \frac{e^{bt}}{G(b)}$$

$$= \frac{e^{bt}}{G(b)} \frac{1}{D^m} 1$$

$$= \frac{e^{bt}}{G(b)} \frac{t^m}{m!}$$

$$= \frac{t^m e^{bt}}{L^{(m)}(b)},$$

since $L^{(m)}(b) \equiv \left(\frac{d^m L(s)}{d s^m}\right)_{s=b}$

$$= (m!\, G(s) + (s-b)P(s))_{s=b}$$

$$= m!\, G(b).$$ (P(s) being some polynomial in s).

Therefore, we have a particular integral

$$y_p(t) = \frac{t^m e^{bt}}{L^{(m)}(b)}.$$

In the case of $m = 0$, the method is well known, but here we have a general method which covers the all cases of m. Now let us see the following examples.

Example 1. $(D^3 + D^2 + D + 1)y(t) = e^{2t}$.

We have $b = 2$, $L(s) = s^3 + s^2 + s + 1$.

Therefore $L(2) = 15$,

$m = 0$,

and $y_p(t) = \frac{e^{2t}}{15}$.

Example 2. $(D^4 - 2D^2 + 1)y(t) = e^t$.

We have $b = 1,\ L(s) = s^4 - 2s^2 + 1$.

Therefore $L(1) = 0$,

$L^{(1)}(1) = (4s^3 - 4s)_{s=1} = 0$,

$L^{(2)}(1) = (12s^2 - 4)_{s=1} = 8$,

$m = 2$,

and $y_p(t) = \dfrac{t^2 e^t}{8}$

Example 3. $(D+1)^5 y(t) = e^{-t}$.

We have $b = -1,\ L(s) = (s+1)^5$.

Therefore $L(-1) = L^{(1)}(-1) = L^{(2)}(-1) = L^{(3)}(-1) = L^{(4)}(-1) = 0$,

$L^{(5)}(-1) = 5! = 120$,

$m = 5$,

and $y_p(t) = \dfrac{t^5 e^{-t}}{120}$.

Example 4. $(D^4 - 1)y(t) = e^{it},\ i = \sqrt{-1}$.

We have $b = i,\ L(s) = s^4 - 1$.

Therefore $L(i) = 0$,

$L^{(1)}(i) = 4i^3 = -4i$,

$m = 1$,

and $y_p(t) = \dfrac{t\,e^{it}}{-4i} = \dfrac{-t\sin t}{4} + \dfrac{it\cos t}{4}$.

Example 5. $(iD^2 + D + 1 - i)y(t) = e^{(1+i)t},\ i = \sqrt{-1}$.

We have $b = 1 + i,\ L(S) = is^2 + s + 1 - i$.

Therefore $L(1+i) = i(1+i)^2 + (1+i) + (1-i) = 0$,

$L^{(1)}(1+i) = (2is + 1)_{s=1+i} = 2i - 1$,

$m = 1$,

and $y_p(t) = \dfrac{t\,e^{(1+i)t}}{2i-1}$.

Although the nonhomogeneous term of the equation in the theorem is the form e^{bt}, the other forms such as $\sin r_1 t$, $\cos r_1 t$, $e^{r_2 t} \sin r_1 t$, $e^{r_2 t} \cos r_1 t$, where r_1, r_2 are real numbers, can also be considered. It should be noted that they can be treated as either real part or imaginary part of e^{bt} with some suitable complex number b.

Furthermore, there is an important application associated with this theorem. Let us consider the problem of a linear differential equation

$$L(D)y(t) = f(t), \qquad (1)$$

where f(t) now is a periodic function. Since a periodic function can always be expressed in a complex form of the Fourier series, here f(t) can often be written as

$$\sum_{\ell=-\infty}^{\infty} c_\ell \, e^{i\ell\omega t}$$

where c_ℓ are complex Fourier coefficients and ω is a given constant angular frequency. Then a particular solution of the equation (1) can be written as

$$y_p(t) = \sum_{\ell=-\infty}^{\infty} \frac{c_\ell \, t^{m_\ell} \, e^{i\ell\omega t}}{L^{(m_\ell)}(i\ell\omega)}, \quad i = \sqrt{-1},$$

where m_ℓ and $L^{(m_\ell)}(i\ell\omega)$ can be obtained according to the rule expressed in the previous theorem. The proof of the above result, which is straight forward, is based on both the theorem given previously and the superposition properties of linear differential equation.