A MEDLEY OF MALAYAN MATHEMATICAL MEMORIES AND MANTISSAE

Richard K. Guy*

I came to Singapore in 1951. Oppenheim was Head of Department, but was on leave, so I was met by Jack Cooke. I think the only members of the department were P.H. Diananda and K.M.R. Menon. In 1952 Eric Milner joined us (and later in the year Malcolm Wicks) and, was sparked by the visit of external examiner Hans Heilbronn, we formed the Malayan Mathematical Society. We soon had many more members from schools, both teachers and students, than the Singapore Mathematical Society has today.There are good reasons for this (you now have many more genuine mathematicians) but I would urge the Society to keep as much contact with the schools as it can. That is where your students come from, and this is where you can influence the importance of early mathematical training of all Singapore's citizens, whether they come to the university or not. I am very pleased to see that the Inter-School Mathematical Competitions continue to be held after all these years; I'm sure that they do good in a number of ways.

The early letters to members from the Secretary of the Society soon formalised themselves into the Bulletin of the Malayan Mathematical Society (B.M.M.S.). Later, largely at Bob Hazell's instigation, this became subtitled, or rather supertitled, Nabla. This name continued for a while after it became the Bulletin of the Singapore Mathematical Society (B.S.M.S.), the predecessor of the Mathematics Medley.

Leong Yu Kiang suggested that I write something for the Medley. These are some of the things that went through my mind, and some of the comments that occurred to me while I was glancing back at earlier issues of the Society's publications.

On page 29 of Vol. 9, No.1 (Feb.1962) of the B.M.M.S. there's an editorial note congratulating Peng Tsu Ann, now head of Department here at N.U.S., on receiving a book prize for the best contribution to the Bulletin in 1961 by a Junior Member. It says "it appears that no [previous] awards were ever made." However, Prof. Teh Hoon Heng, who has been Head of Department at both Nanyang University and N.U.S., recently showed me the book prize that he was awarded for best contribution by a Junior member in 1955. Even if the announcement wasn't published in the Bulletin, I'm sure it must at least be in the Minutes. Louis Chen is bravely fighting the termites in an effort to stop these being consumed; I hope he is successful!

In the seventies the editors seem to have been short of material, since there was duplication of earlier articles. Perhaps the subject matter was so good that

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it was worth repeating! For example, in B.S.M.S. Dec. 1972, 45-51, there's a proof of J.S. Samy's conjecture (B.M.M.S. 5 (1958) 17) concerning the number of 3 x 3 magic squares with positive integer entries and central element n. Samy's combinatorial intuition was that, modulo 6, the number could be expressed as an exercise in linear programming, giving the formula in the form

$$2(n-1)(n-3) + 2\lfloor n/2 \rfloor - 2\lfloor (n-1)/2 \rfloor - 8\lfloor (n-1)/3 \rfloor$$

in the same year as the conjecture (B.M.M.S. *5*(1958) 87 – 89). As I write this it occurs to me that a neater form for the formula is

$$8\lfloor (3n^2 - 16n + 24)/12 \rfloor$$

Other articles, in B.S.M.S., 1973, 11 – 12 and 1974, 23 – 24, had appeared before in B.M.M.S., *9*(1962) 77 and 110 – 111.

I'll next comment on some problems, solutions to which appeared in B.S.M.S. *11*, No. 1 (June, 1969) 42 – 44. Peng Tsu Ann gave a trigonometric solution to P. 3/66, in which A. Oppenheim asked to show that the maximum value of

$$\cos\theta_1 + \cos\theta_2 + \dots + \cos\theta_n \qquad (1)$$

is $n\cos(\pi/n)$, where $\theta_1, \theta_2, \dots, \theta_n$ are positive acute angles with sum π. Here is a simple geometrical solution of a slightly more general result.

Imagine a chain of unit rods, laid out as in Figure 1, so that successive tods make positive acute angles $\theta_1, \theta_2, \dots, \theta_n$ with the x-axis. Then the expression (1) is the length of the projection on the x-axis, which is a maximum when the chain is pulled out into a straight line, with all angles equal. If the sum of the angles is fixed, s say, then the maximum is $n\cos(s/n)$, where s may be any number, $0 < s < n\pi/2$.

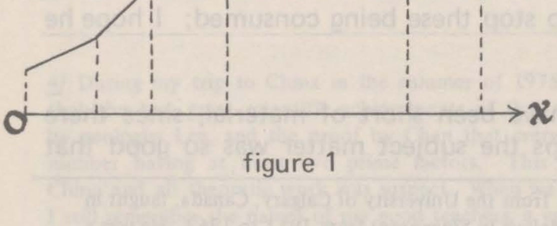

figure 1

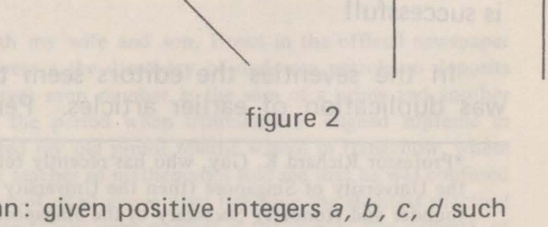

figure 2

Problem 4/66 was set by Peng Tsu Ann: given positive integers a, b, c, d such that $a^2 > 2b^2 > 2d^2 > c^2$; using only properties of the integers, show that $(a-c)^2 > 2(b-d)^2$. P. H. Diananda's solution doesn't explicitly use properties of the integers, and the result is true for positive real numbers. However, if they are integers we can prove a little more. Draw concentric squares of sides $a\sqrt{2}$, $b\sqrt{2}$,

$d\sqrt{2}$, $c\sqrt{2}$, the first and last with parallel sides, the others making angles $\pi/4$ with them (Figure 2). A, B, D, C are corners of these squares. The strict inequalities demand that the C square is inside the A square, and that corners B, D lie between them. $AC = a - c$, $BD = b - d$ and $AC > BD\sqrt{2}$, so the inequality follows. If a, b, c, d are integers, then $BD\sqrt{2}$ is not, and $(a-c)^2 \geqslant 2(b-d)^2 + 1$. To see that this is best possible, take $a = 10$, $b = c = 7$, $d = 5$.

There is an unanswered challenge in P. 5/66, proposed and solved by Malcolm Wicks: find the number of different "k-deals" from a pack of cards consisting of M (different) red cards and N blue ones, where a k-deal contains k more red cards than blue. The answer, "$M + N$ choose $N + k$" strongly suggests that there is a simple combinatorial argument. When I first drafted this article, I couldn't see it. But after sleeping on it (the bed and the bathroom are good places for doing mathematics), I realized that there was indeed such an argument.

A k-deal contains $k + r$ red cards and r blue ones for some r, $0 \leqslant r \leqslant \min(M - k, N)$. For every choice of r blue cards there is a corresponding complementary set of $N - r$ blue cards that are *not* chosen. So every k-deal corresponds to a choice of $k + r$ red cards and (the complement of a choice of) $N - r$ blue ones, i.e. $N + k$ cards altogether. But every one of the $\binom{M+N}{N+k}$ choices of $N + k$ cards from the pack of $M + N$ will have $k + r$ red cards and $N - r$ blue ones, for some r, $0 \leqslant r \leqslant \min(M - k, N)$. So these are equinumerous with the number of k-deals(take the r blue cards which *haven't* been dealt, with the $k + r$ which *have*).

Symmetrically, since $\binom{M+N}{N+k} = \binom{M+N}{M-k}$, we can argue with red and blue cards interchanged. Any choice of $M - k$ cards contain b blue ones, for some b, $0 \leqslant b \leqslant \min(M - k, N)$, and hence $M - k - b$ red ones. The corresponding k-deal consists of the b blue cards and the $k + b$ red cards not chosen.

I'll conclude with some remarks about the ubiquitous *Catalan numbers.* On page 108 of B.M.M.S. *7*, No. 3 (Oct. 1960) I persuaded Eric Milner, against his better judgement (there is an apology in Note 110 on page 68 of B.M.M.S. *8*, No. 2 (Apr. 1961), to publish a phoney combinatorial solution to the first part of one of the Erdos-Posa problems. On the same page it was stated that we didn't have a combinatorial solution to the second part.

Here is the problem: if $A_n = \binom{2n}{n}/(n+1)$

(i) show that A_n is an integer,

(ii) show that $A_n = A_o A_{n-1} + A_1 A_{n-2} + \ldots + A_{n-1} A_o$ (2)

and here are two combinatorial solutions to each part. One part of one solution had already appeared in a famous paper (Note go in B.M.M.S. *5*, No. 45 (Aug. 1958) 57 — 60) which anticipated papers of John Moon & Leo Moser (Canad. Math. Bull. *6*(1963) 175 — 178, and of Hans Rademacher (Illinois J. Math. *9*(1965) 361 — 380). This was reprinted as the University of Calgary's "Yellow Series", No. 9, together with an amplified version of Will Brown's excellent bibliography (Amer. Math. Monthly, *72*(1965) 973 — 977); write to the U. of C. if you want a copy.

In each solution we define A_n as a counting function, so that it is necessarily an integer, and we show that it satisfies the convolutionary recurrence (2). We conclude, in each case, by showing that our A_n satisfies the further relation

$$(n+1)\,A_n = 2(2n-1)\,A_{n-1} \tag{3}$$

repeated application of which gives

$$n!(n+1)!\,A_n = (2n)!\,A_o$$

In the original definition, and in each of ours, it is appropriate to take $A_o = 1$, so the identity of the various A_n is established.

First define A_n as the number of different ways of dissecting a polygon with $n + 2$ sides into n triangles by joining $n - 1$ pairs of vertices with non-intersecting diagonals. To make it clear what is meant by "different", we eliminate any questions of symmetry by specializing one side of the $(n + 2)$–gon as a "root" (shown thick in Figures 3 to 6). This definition, with $A_o = 1$, gives $A_1 = 1$ (triangle), $A_2 = 2$ (draw either diagonal of a quadrilateral), $A_3 = 5$ (Figure 3) and $A_4 = 14$ (In Figure 4a there are six vertices at which the three diagonals may meet; in 4b the long diagonal can be any one of three, and the pattern of diagonals can form either a letter N or its reflexion; in 4c the inside triangle can have either the odd or the even numbered vertices of the hexagon for its vertices; 6 + (2 x 3) + 2 = 14).

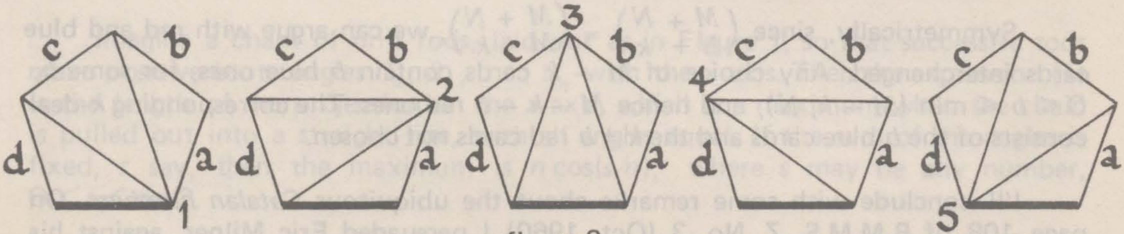

figure 3

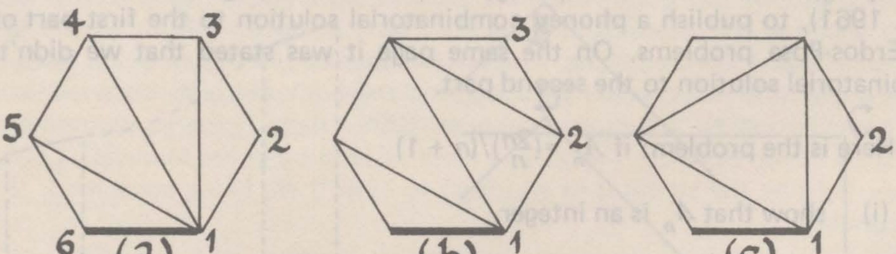

figure 4

To establish the convolution (2), notice that the root side belongs to a unique triangle of the dissection (shown shaded in Figure 5). This triangle partitions the original $(n + 2)$–gon into an $(r + 2)$–gon and an $(n - r + 1)$–gon, for one of the n possible values of r, $0 \leqslant r \leqslant n - 1$, i.e. the third vertex of the shaded triangle may be any one of the vertices of the $(n + 2)$–gon other than the 2 ends of the root.

Note that for the extreme values $r = 0$ and $r = n - 1$, one of the two subpolygons is a degenerate 2–gon, but that we have defined $A_o = 1$. The numbers of different dissections of the $(r + 2)$–gon and the $(n - r + 1)$ gon are, by definition, A_r and A_{n-r-1} so

$$A_n = \sum_{r=0}^{n-1} A_r A_{n-r-1}$$

which is (2).

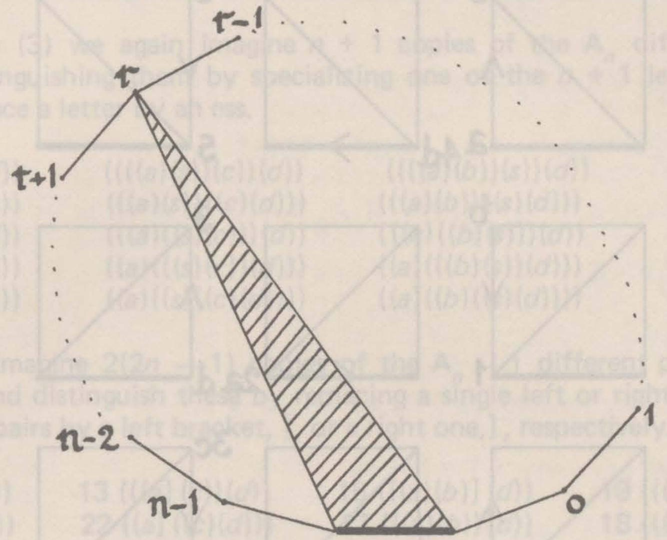

figure 5

To establish (3) we draw (in our imagination) $n + 1$ copies of each of the A_n dissections of an $(n + 2)$–gon, and distinguish them by marking one side, other than the root, in each possible way. For example, imagine $n + 1 = 4$ copies of Figure 3, with side *a* marked in the first, *b* in the second, *c* in the third and *d* in the fourth.

Also draw $2(2n - 1)$ copies of each of the A_{n-1} dissections of an $(n + 1)$ –gon, and distinguish these by drawing an arrow on one edge of each dissection. There are $n + 1$ side (we include the root this time) and $n - 2$ diagonals, $2n - 1$ edges in all, and the arrows can be drawn in either direction, so they just suffice to distinguish the $2(2n - 1)$ copies. We have done this for $n = 3$ in Figure 6.

It remains to establish a one-one correspondence between the $(n + 1)A_n$ diagrams and the $2(2n - 1)A_{n-1}$ diagrams. To go from Figure 3 to Figure 6, collapse the triangle containing the marked side by identifying the points of this side, leaving a single edge, and mark it with an arrow pointing from the collapsed side to the opposite vertex.

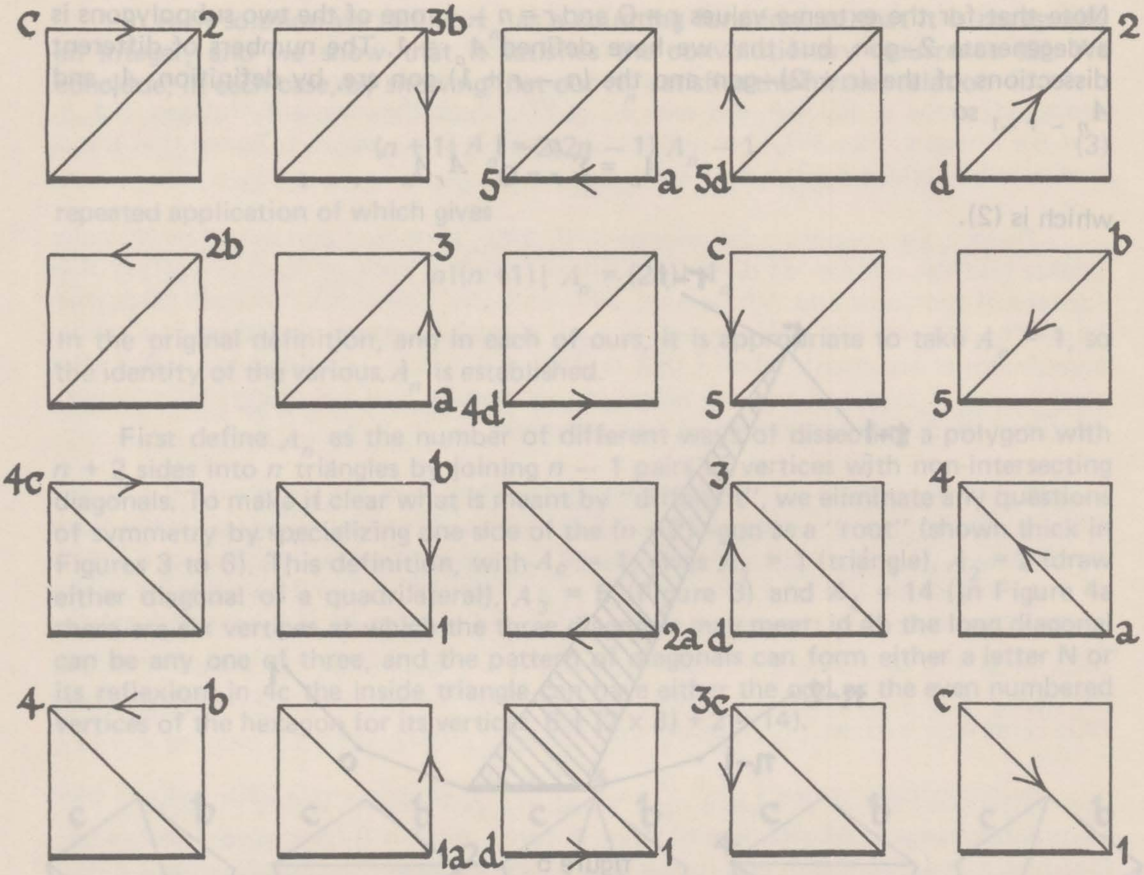

Figure 6

To go from Figure 6 to Figure 3, widen the edge marked with an arrow into a dart-shaped triangle, one of whose vertices is the head of the arrow, and the opposite side, a new $(n + 2)$th side of the polygon, corresponds to the tail of the arrow. The labelling of Figure 6 corresponds to that of Figure 3, a lettered vertex (the tail of an arrow) corresponding to a collapsed marked side.

The second solution derives from the best known manifestation of the Catalan numbers. If we have a string of $n + 1$ letters with the implication that they are to be multiplied together, then the result may depend on the order in which the n multiplications are performed, if multiplication is not associative. In how many ways can we insert n pairs of parentheses to indicate the order? We will take this as a new definition of A_n. For example, $A_3 = 5$:

$(((ab)c)d)$ $((ab)(cd))$ $((a(bc))d)$ $(a((bc)d))$ $(a(b(cd)))$

To clarify our second pair of proofs, enclose each of the $n + 1$ individual letters in its own pair of parentheses, so that we now have $2n + 1$ pairs of parentheses, and each expression is now in the shape

$$((\ldots r+1\ldots)(\ldots n-r\ldots))$$

where the two inner pairs of parentheses contain $r + 1$ letters and $n - r$ letters, for some r, $0 \leqslant r \leqslant n-1$. So we again have

$$A_n = \sum_{r=0}^{n-1} A_r A_{(n-r-1)}$$

which is (2)

To prove (3) we again imagine $n + 1$ copies of the A_n different parenthesizations, distinguishing them by specializing one of the $n + 1$ letters. In the example we replace a letter by an ess.

(((*s*)(*b*))(*c*))(*d*))	(((*a*)(*s*))(*c*))(*d*))	(((*a*)(*b*))(*s*))(*d*))	(((*a*)(*b*))(*c*))(*s*))
(((*s*)(*b*))((*c*)(*d*)))	(((*a*)(*s*))((*c*)(*d*)))	(((*a*)(*b*))((*s*)(*d*)))	(((*a*)(*b*))((*c*)(*s*)))
(((*s*)((*b*)(*c*)))(*d*))	(((*a*)((*s*)(*c*)))(*d*))	(((*a*)((*b*)(*s*)))(*d*))	(((*a*)((*b*)(*c*)))(*s*))
((*s*)(((*b*)(*c*))(*d*)))	((*a*)(((*s*)(*c*))(*d*)))	((*a*)(((*b*)(*s*))(*d*)))	((*a*)(((*b*)(*c*))(*s*)))
((*s*)((*b*)((*c*)(*d*))))	((*a*)((*s*)((*c*)(*d*))))	((*a*)((*b*)((*s*)(*d*))))	((*a*)((*b*)((*c*)(*s*))))

We also imagine $2(2n - 1)$ copies of the A_{n-1} different parenthesizations of n letters, and distinguish these by replacing a single left or right parenthesis out of the $2n - 1$ pairs by a left bracket, [, or a right one,], respectively.

12 (([*b*)(*c*))(*d*))	13 (((*a*] (*c*))(*d*))	16 (((*a*)(*b*)] (*d*))	19 (((*a*)(*b*))(*c*)]
21 ([*b*)((*c*)(*d*)))	22 ((*a*] ((*c*)(*d*)))	17 (((*a*)(*b*))[*d*))	18 (((*a*)(*b*))(*c*])
11 ([*b*)(*c*))(*d*))	14 (((*a*)[*c*))(*d*))	15 (((*a*)(*b*])(*d*))	29 ((*a*)((*b*)(*c*))]
10 [((*b*)(*c*))(*d*))	24 ((*a*)([*c*)(*d*)))	25 ((*a*)((*b*] (*d*)))	28 ((*a*)((*b*)(*c*)])
20 [(*b*)((*c*)(*d*)))	23 ((*a*)[(*c*)(*d*)))	26 ((*a*)((*b*)[*d*)))	27 ((*a*)((*b*)(*c*]))

To see the one-one correspondence, notice that the special letter is enclosed in parentheses which are nested in other parentheses in one of the forms.

((*s*)(. . .))   or   ((. . .)(*s*))

and these correspond to

[. . .)   or   (. . .]

respectively. In the example, $n = 3$, $A_{n-1} = 2$, i.e. there are just two different parenthesizations of three letters. These are usually written

1. $(xy)z$ 2. $x(yz)$

but we have decorated them with $2n - 1 = 5$ pairs of parentheses. Think of the 10 individual parentheses as being labelled from left to right with the digits 0 to 9. The two-digit numbers on the left of the expressions in the second array run from 10 to 29. The first digit indicates which of the two parenthesizations is represented, and the second is the label corresponding to the bracket, [or], which replaces one of the 10 parentheses. The two displays are laid out with corresponding expressions in corresponding places.

Combinatorics hasn't quite acquired the respectability that other branches of mathematics sometimes assume, and some people regard proofs of this kind as conjuring tricks, designed to delude the observer into thinking that something has happened which hasn't really taken place. So, for the infidels, here is how Euler might have connected the various formulas (compare B.M.M.S. *5*(1958) 58 and *7* (1960) 108).

We take (2), with $A_0 = 1$, as our *definition* of A_n and find its *generating function*,

$$y = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots$$

$$= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + \ldots$$

$$y^2 = A_0 A_0 + (A_0 A_1 + A_1 A_0) x + (A_0 A_2 + A_1 A_1 + A_2 A_0) x^2 + \ldots$$

$$= A_1 + A_2 x + A_3 x^2 + \ldots$$

by definition. So

$$1 + xy^2 = 1 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots = y$$

$$4x^2 y^2 - 4xy + 1 = 1 - 4x$$

$$2xy - 1 = -\sqrt{1-4x}$$

where we choose the negative root, since we know that $y(o) = 1$. Notice that this is just formal manipulation and that there is no need to worry about such things as convergence, but those who like to worry may take $|x| < 1/4$.

$$y = (1 - \sqrt{1-4x})/2x$$

By the binomial theorem,

$$y = (-1/2x)\, \Sigma_{n=0} \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \ldots \left(-\frac{2n-1}{2}\right) (-4x)^{(n-1)} \Big/ (n+1)!$$

$$= \Sigma_{n=0}\, 1.3.5 \ldots (2n-1) 2^n x^n / (n+1)!$$

$$= \Sigma_{n=0}\, (2n)! x^n / n!(n+1)!$$

showing that $A_n = (2n)!/n!(n+1)!$ as originally defined. Our definition, using (2), clearly shows that A_n is an integer. A simple way to see that this is so from the original definition is to note that

$$A_n = (2n+1) A_n - 2n A_n = \binom{2n+1}{n+1} - 2\binom{2n}{n+1}$$

and to use the fact that the binomial coefficients are integers.

It has been most gratifying to be in Singapore once again and to see that not only is the young island republic making remarkable progress in a great variety of ways, but that Mathematics in particular is one of the more lively and flourishing aspects. I shall certainly come back, and I shan't wait another 22 years before the next time!