FROM EUCLIDEAN GEOMETRY TO TRANSFORMATION GEOMETRY

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§ 1. Introduction

Since the early seventies, the geometry part of the high school mathematics curriculum in this region has included topics like vectors, reflections, translations, rotations and symmetries to replace part of the more traditional geometry curriculum. Transformation geometry has thus been included in the "modern mathematics" syllabus. The approach adopted for the study of geometry is different from the traditional one. Geometrical proofs of the old have given way to activity-oriented proof, using the ideas of symmetry, translation, reflection, rotation and enlargement [6].

The switch to the new syllabus has led some of us to ask whether this should be the type of geometry taught in the secondary schools. Many of our mathematics teachers have been trained in and are more familiar with the usual Euclidean deductive approach in teaching Euclidean plane geometry. What sort of geometry should be taught in the secondary schools remains a controversy.

The purpose of this note is to relate the classical Euclidean geometry and the more recent transformation geometry. Before we do this, let us describe briefly some historical facts and the nature of the two geometries.

§ 2. Euclidean Plane Geometry

The original (incomplete) axiomatic and deductive geometry was recorded in Euclid's lecture notes "Elements" two thousand years ago. After the discovery of the so-called non-Euclidean geometry a century and a half ago, the original Euclidean geometry was later reorganised by a number of mathematicians in the early twentieth century, notably by Moritz Pasch, Giuseppi Peano, Mario Pieri, David Hilbert and G. D. Birkhoff. The (deductive) plane geometry in the traditional mathematics curriculum follows the approach of Euclid and Hilbert. Basically plane geometry is treated as a deductive system, using the natural geometry on the plane as a model.

The most important axiom that governs the geometric part of the deductive system is the congruence postulate which says that two triangles are congruent if the two sides and the included angle of one triangle are congruent to two sides and the included angle of the other triangle, whereas the parallel postulate guarantees that the plane is Euclidean!

Let us state the Euclid parallel postulate.

"If a transversal falls on two lines in such a way that the interior angles on one side of the transversal are less than two right angles, then the lines must meet on that side on which the angles are less than two right angles"

This postulate is equivalent to the following Playfair's parallel postulate:

" Through a point not on a given line, exactly one line can be drawn in the plane parallel to the given line. "

It is from the parallel postulate that we can prove theorems like those which state that the sum of the interior angles of a triangle is 180° and that the sum of the interior angles of a quadrilateral is 360°. Theorems of this nature characterize the fact that the underlying space is plane.

On the other hand, let us modify the parallel postulate as follows:

"Through a given point C , not on a given line AB, there pass at least two lines which do not intersect the given line."

Then we shall get the hyperbolic geometry where the sum of the interior angles of a triangle is less than 180°.

If we change the parallel postulate to:

"Through a given point not on a given line, there is no line through the given point parallel to the given line."

Then we would get the elliptic geometry, the kind of geometry on a sphere where any two "lines" intersect and the sum of the interior angles of a triangle is greater than 180°.

From Euclid's parallel postulate, we obtain only some qualitative data about the plane but we cannot say anything about the size or shape of an object in the plane. To be able to do this we need the SAS congruence postulate which states that

"Given two triangles Δ ABC and Δ A'B'C', if the congruences AB=A'B', AC=A'C' and $\measuredangle$BAC = $\measuredangle$B'A'C' hold then the congruences $\measuredangle$ ABC = $\measuredangle$A'B'C', $\measuredangle$ ACB = $\measuredangle$A'C'B' and BC = B'C' hold."

It is this postulate that leads to some of the beautiful results in plane geometry. For example we have the following.

Theorem SAS congruence condition is equivalent to SSS congruence condition.

Theorem The locus of all points equidistant from two points M & N is the perpendicular bisector of MN.

Theorem In a triangle ABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

We would not give details here. For a detailed development of (deductive) Euclidean plane geometry, the reader is referred to [2] .

§ 3. From Euclidean Geometry to Transormation Geometry

After M. Pasch and many others had systematically reorganized Euclidean geometry along the spirit of Euclid's "Elements", Mario Pieri adopted a quite different approach in a study of Euclidean geometry in 1899. He considered the subject of his study to be an aggregate of undefined elements called "points" and an undefined concept called "motion". For example, a straight line AB is the set of points which remain fixed under any effective motion which leaves A and B fixed.

Although Pieri's treatment of Euclidean geometry was not widely accepted, the development of certain modern concepts is apparent. We have for instance, the idea of transformation as a mapping. Pieri's motions are the so-called direct isometries

$$\begin{aligned} x' &= ax + by + h \\ y' &= cx + dy + k \end{aligned} \qquad \text{where } \det \begin{pmatrix} a, b \\ c, d \end{pmatrix} = 1$$

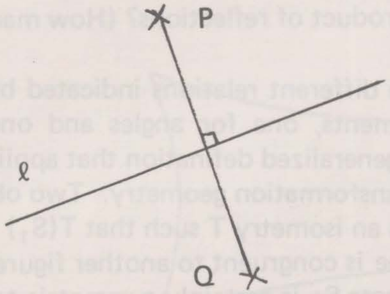
He considered Euclidean goemetry as the study of the properties and relations of configurations of points which remain invariant under the group of direct isometries. This idea had earlier (in 1972) been generalized, in the famed Erlanger Programm, to form the basis of Felex Klein's remarkable codification of geometries. According to Klein, a geometry (not necessarily Euclidean goemetry) can be characterized by a group of transformations and the definitions and theorems of the geometry are simply the invariants, invariant properties and invariant relations under the group of actions.

Thus the study of geometry, as an empirical experience and later as a deductive system, has been reduced to the algebra of group actions on the underlying space. We shall discuss briefly in the next section some ideas and concept in transformation geometry.

§ 4. Transformation Geometry

Let us recall some of the basic concepts in transformation geometry. A reflection

R_ℓ in a line ℓ is the mapping defined by



$$R_\ell(P) = \begin{cases} P & \text{, if } P \in \ell \\ Q & \text{, if } P \notin \ell \text{ and} \\ & \ell \text{ is the perpendicular bisector} \\ & \text{of the line } PQ. \end{cases}$$

An isometry of the plane R^2 is a transformation (one to one, onto mapping) which preserves distances. Reflections and translations are isometries of the plane. It is easy to see the set of isometries form a group [4]. The next question is whether the set of reflections and their products form a group. The answer is "yes". In fact the group of all reflections and their products is the same as the group of all isometries.

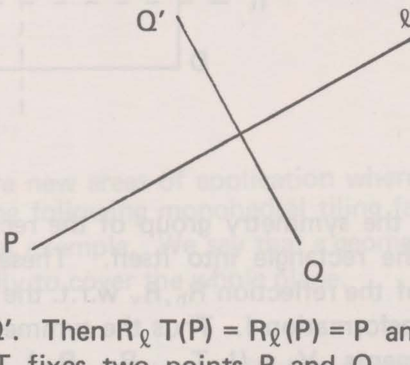
Theorem: A product of reflections is an isometry. Every isometry is a product of at most three reflections.

Proof: It is clear that a product of reflections is an isometry.

Now we want to prove the second statement in the theorem.

If the isometry fixes two points or more, then it can be easily shown to be either an identity or a reflection.

We then show that an isometry T that fixes exactly one point P is a product of two reflections. Let Q be a point different from P. Let $Q' = T(Q)$ and let ℓ be the perpendicular bisector of QQ'. Since T is an isometry, $T(P)=P$ and $T(Q)=Q'$, we have $PQ = PQ'$ and P is a point on the line ℓ.



Therefore $R_\ell(P) = P$ and $R_\ell(Q) = Q'$. Then $R_\ell T(P) = R_\ell(P) = P$ and $R_\ell T(Q) = R_\ell(Q') = Q$. Thus the isometry $R_\ell T$ fixes two points P and Q. Hence $R_\ell T$ is the identity transformation or a reflection. However $R_\ell T$ cannot be the identity because this would imply that $R_\ell R_\ell T = R_\ell$, that is, $T = R_\ell$, a reflection that fixed more than one point. Hence $R_\ell T$ is a reflection, that is, $R_\ell T = R_m$ for some line m. With this $T = R_\ell R_m$ a product of two reflections.

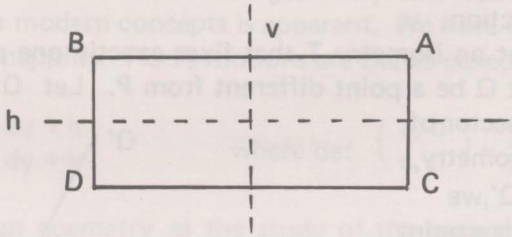
Now, for an arbitrary isometry T, let us suppose T sends point P to a different point Q. Let m be the perpendicular bisector of PQ. Then $R_m T$ fixes P. Thus $R_m T$ must be a product of at most 2 reflections. If $R_m T$ is the product of R_{ℓ_1} and R_{ℓ_2}, that is, $R_m T = R_{\ell_1} R_{\ell_2}$. Then $T = R_m R_{\ell_1} R_{\ell_2}$, a product of three reflections.

A translation of the plane is also an isometry. Perhaps the reader should try to convince himself/herself that a translation is a product of reflections? (How many?)

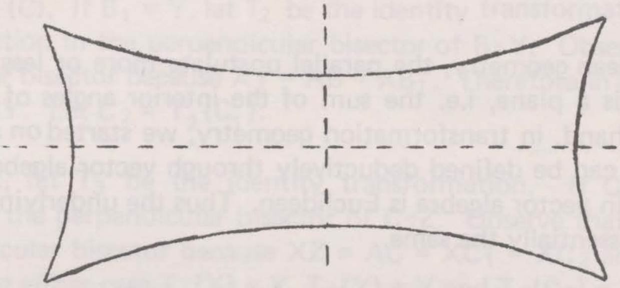
In Euclidean plane geometry, there are three different relations indicated by the same words "is congruent to", one for segments, one for angles and one for triangles. All three can be combined under a generalized defination that applies to arbitrary geometrical objects in the plane in transformation geometry. Two objects S_1 and S_2 are said to be congruent if three exists an isometry T such that $T(S_1) = S_2$. For example, a geometric figure S_1 in the plane is congruent to another figure S_2 if there exists R_ℓ such that $S_2 = R_\ell(S_1)$. In this case S_1 is certainly symmetric to S_2. The concept of symmetry plays an important part in transformation geometry.

We can now answer the question, "what is transformation geometry?". It is the study of invariants, invariant properties and invariant relations in the plane under the group of isometries. Since isometries preserve, for example, betweenness, mid-points, segments, rays, triangles, rectangles, conic sections, angles and perpendicularity, these form the subjects of study in transformation geometry.

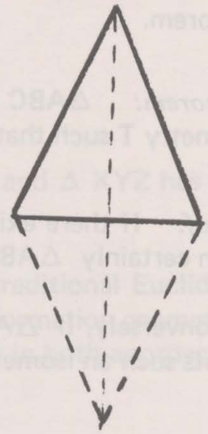
For a particular object, say a rectangle ABCD which is not a square. The question here is: What are the properties of the rectangle that are of geometric importance?



We look at the symmetry group of the rectangle, namely all the isometries which transform the rectangle into itself. These are the symmetries of the rectangle. It consists of the reflection R_h, R_v w.r.t. the two axes, the 180° rotation T_o and the identity transformation I. Thus the symmetry group of rectangle ABCD is a group of four elements $V_4 = (I, T_o, R_h, R_v)$. The geometry of the rectangle ABCD comprises invariant relations under the actions of the group V_4. From the group of symmetries, we know that the size of an interior angle remains invariant (that is, all four interior angles are the same). From this, we see that the four interior angles being equal is a goemetrical property of the rectangle. In fact, the following figure has the same group of symmetry. The reader could determine some of its geometric properties.



In general, the symmetries of a set S in the plane form a group called the symmetry group of S. And the geometry of the set S is the study of the group actions of the symmetry group acting on S.



Let us look at the geometry of an (non-equilateral) isosceles $\triangle ABC$, where AB = AC. One of the known geometric facts $\triangle ABC$ is that $\angle B = \angle C$. To be able to say this, we must first look at the symmetry group $\triangle ABC$. We first reflect $\triangle ABC$ to $\triangle A'BC$. Then AA' is perpendicular to BC. Now it is clear that the symmetry group of $\triangle ABC$ is $(I, R_{AA'})$. Since $\angle B$ and $\angle C$ are the same under the actions of the group, the fact that $\angle B = \angle C$ is a geometrical property attached to $\triangle ABC$.

In transformation geometry, there are new areas of application where traditional Euclidean geometry did not cover. The following monohedral tiling fact from the tiles manufacturing industry is a typical example. We say that a geometric figure F tiles the plane if F can be used repeatedly to cover the whole plane.

Theorem : Any triangle tiles the plane; any quadrilateral tiles the plane. A hexagon with a point of symmetry tiles the plane.

For a proof, please refer to [5].

§ 5. The Relationship Between Traditional Euclidean Geometry and Transformation Geometry

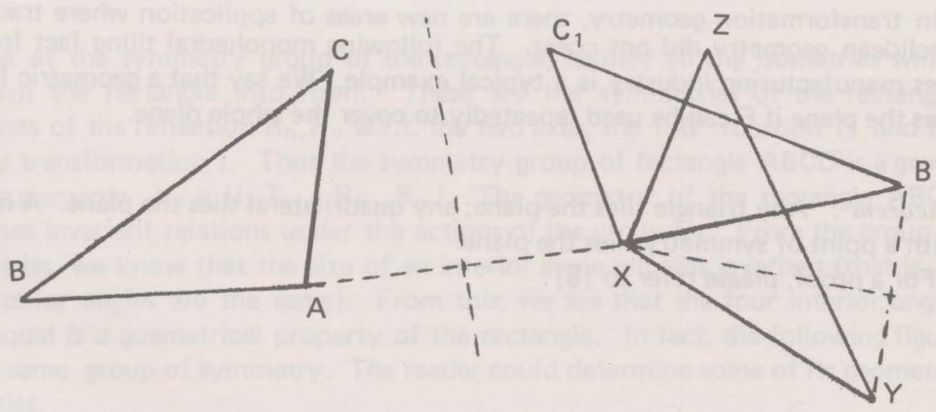
In traditional Euclidean geometry, the parallel postulate more or less defines that the underlying space is a plane, i.e. the sum of the interior angles of a triangle is 180°. On the other hand, in transformation geometry, we started on an Cartesian plane, and this plane can be defined deductively through vector algebra [1]. The parallel concept used in vector algebra is Euclidean. Thus the underlying plane used in both geometries is essentially the same.

As we have mentioned in § 2, the most important geometric tool used in developing the traditional Euclidean goemetry on the plane is the congruence axiom, namely, two triangles are congruent if they satisfy the SAS criterion. Now, in transformation goemetry, two triangles Δ_1 and Δ_2 are congruent if there exists an isometry T (which is the product of at most three reflections) such that $T(\Delta_1) = \Delta_2$. These two congruence concepts mean the same thing as in the following theorem.

Theorem: ΔABC and ΔXYZ have the SAS property if and only if there exists an isometry T such that T(A) = X, T(B) = Y and T(C)= Z.

Proof: If there exists an isometry T such that T(A) = X, T(B) = Y and T(C) = Z, then certainly ΔABC and ΔXYZ has the SAS property.

Conversely, if ΔABC and ΔXYZ has the SAS property, we shall show that there exists such an isometry T.



If $X = A$, take T_1 to be the identity transformation. If $X \neq A$, let T_1 be the reflection in the perpendicular bisector of AX. In either case $T_1(A) = X$. Let $B_1 = T_1(B)$ and $C_1 = T_1(C)$. If $B_1 = Y$, let T_2 be the identity transformation. If $B_1 \neq Y$, let T_2 be the reflection in the perpendicular bisector of B_1Y. Observe that X lies on this perpendicular bisector because $XY = AB = XB_1$. Therefore in either case, $T_2(X) = X$, $T_2(B_1) = Y$. Let $C_2 = T_2(C_1)$.

If $C_2 = Z$, let T_3 be the identity transformation. If $C_2 \neq Z$, let T_3 be the reflection in the perpendicular bisector of C_2Z. Observe that the line XY is in fact the perpendicular bisector because $XZ = AC = XC_1 = XC_2$ and $YZ = BC = B_1C_1 = YC_2$. Thus in either case $T_3(X) = X$, $T_3(Y) = Y$ and $T_3(C_2) = Z$.

Now let $T = T_3T_2T_1$. Then

$$T(A) = T_3T_2T_1(A) = T_3T_2(X) = T_3(X) = X,$$

$$T(B) = T_3T_2T_1(B) = T_3T_2(B_1) = T_3(Y) = Y,$$

and $$T(C) = T_3T_2T_1(C) = T_3T_2(C_1) = T_3(C_2) = Z.$$

Q.E.D.

In proof of the above theorem, we used the fact that ΔABC and ΔXYZ has the SSS property which is equivalent to SAS.

From the above theorem, we see that congruence axiom in traditional Euclidean goemetry is equivalent to the reflection and symmetry in transformation geometry. Hence we conclude that it is essentially the same plane goemetry in both approaches but the two approaches are quite different in spirit.

§ 6. Teaching Transformation Geometry

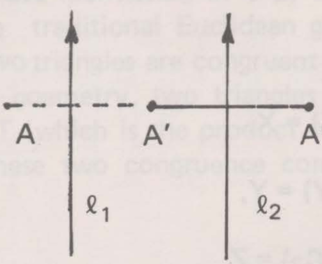
Students are learning modern concepts like translation or rotation in high schools today. However many teachers may not see why such ideas in transformation geometry are taught and may even feel that students are learning a topic that leads to nowhere. The importance of geometry can be seen in the work of many factory workers such as fitters, carpenters and others in engineering work. It is interesting to see how a construction worker determines the corner of a house or to watch how a worker cuts out the exact length to fit certain engineering need. The importance of geometry calls for the teaching of the subject in a proper manner.

In teaching transformation geometry, the teacher must not forget to discuss ordinary symmetries that occur in real life. After some knowledge in goemetry, a very good way to introduce the basic concepts in transformation geometry, perhaps in Form Three, is to begin with the symmetry and homogenity of a plane. From the symmetry of a plane we can define reflection for the plane. In fact, it is due to the fact that the plane is symmetric about any line that we are able to define reflection in a given line. By homogenity of the plane is meant the geometry at and around a point is the same as that at and around any other point.

Concepts like reflection, rotation and translation are being introduced in Forms Two and Three, but very seldom do we find that these concepts are linked together. For example, we have the following important results.

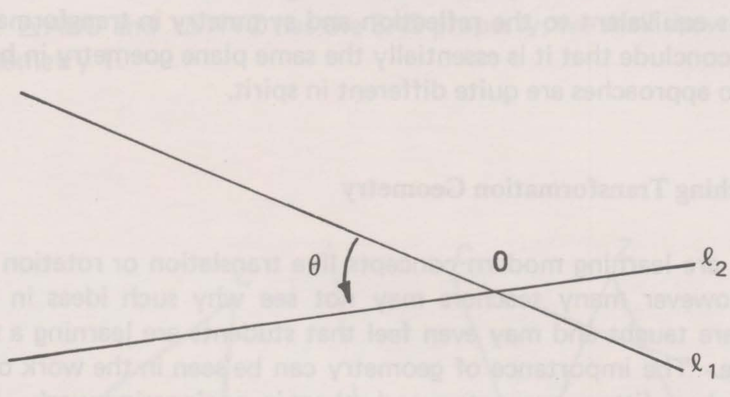
Theorem A

If two straight lines ℓ_1, and ℓ_2 are parallel and are at a distance of h from one another, then a translation of 2h in the direction perpendicular to the two parallel lines can be regarded as the result of reflection in line ℓ_1 followed by reflection in line ℓ_2.



Theorem B

If two straight lines ℓ_1 and ℓ_2 intersect at an angle θ then reflection in ℓ_1 followed by reflection in ℓ_2 results in a rotation of angle 2θ about the point of intersection 0.



Any rotation about a point is the result of combining two such reflections.

The above two results should also help students in understanding the composition of two transformations.

After rotation, translation and reflection have been linked we could then go ahead with determining other elementary goemetrical facts. What follows then would not be very different from what is done in Euclidean geometry.

§ 7. Conclusion

Formal deductive proofs in transformation geometry can be made more convincing with the help of paper folding. Use of paper folding in transformation geometry would also enhance the learning of geometrical ideas in students. Via paper folding and geometrical construction, geometry would appear "real" to the students. Geometrical construction appears to be neglected in schools perhaps partly because of the examination system. But this topic may be very useful to those who may not do well academically in school and later drop out of school to pick up some technical skills. Moreover, geometrical construction would provide training for students to use intuition, imagination and geometrical pattern in their thinking. Transformation geometry provides a good link between geometry and algebra. Earlier exposure to well-taught transformation geometry would also prepare students well for ideas and concepts in later study of mathematics.

References

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