SOLUTIONS TO PART A

1. We have \((13)^{62} = (169)^{30} = (169)^{(169^2)^{15}}\).
   The unit digit of \((169)^2\) is 1 and so the unit digit of \((169^2)^{15}\) is also 1. Hence the unit digit of \((13)^{62}\) is 9. Hence the correct answer is (d).

2. As \(\log_{10} x + \log_{10} y = 2\), we have \(\log_{10} xy = 2\). Hence \(xy = 100\). Now \(1/x + 1/y \geq 2\sqrt{(1/xy)} = 1/5\). However, when \(x = y = 10\), \(1/x + 1/y = 1/5\). Hence the minimum value of \(1/x + 1/y\) is 1/5.
   The correct answer is therefore (d).

3. We have \(f(f(x)) = \left[\frac{a(ax/(bx + 1))}{b(ax/(bx + 1)) + 1}\right]
   = \frac{a^2 x}{abx + bx + 1}
   = \frac{a^2 x}{(ab + b)x + 1}.
   Hence if \(f(f(x)) = x\), we have \(ab + b = 0\) and \(a^2 = 1\). From this we see that one feasible solution is \(a = -1\) and \(b\) can be arbitrary. So the correct answer is (c).

4. Only the seven digits 0, 1, 2, 4, 6, 8 and 9 can be used to form positive integers less than 10,000. Thus there are \(7^3\) such integers ending with 1, 2, 4, 6, 8 and 9 respectively. Hence the sum of the unit digits of all these integers is equal to \(7^3 (1 + 2 + 4 + 6 + 8 + 9) = 10290\). Similarly, the sums of the tenth, hundredth, and thousandth digits of these integers are respectively equal to \(10290 \times 10\), \(10290 \times 100\) and \(10290 \times 1000\). Hence the required sum is equal to

\[
10290 (1 + 10 + 100 + 1000) = 11432190.
\]
   The correct answer is thus (a).

5. Extent BD to intersect the circle at E and also extend OD in both directions, intersecting the circle at F and G as shown in the following figure.

We then have \((BC)(BE) = (AB)^2\) from which we get \(4(DE + BD) = 64\) and so \(4(DE + 8) = 64\). Hence \(DE = 8\). Also, \((DE)(DC) = (DF)(DG)\). Thus, \(8 \times 4 = (r - 3)(r + 3)\) where \(r\) is the radius of the circle, from which we get \(r^2 = 41\).
   Hence the correct answer is (e).
6. From the figure below, we see that:

\[
\frac{S(\triangle AOC)}{S(\triangle ABC)} = \frac{AF}{AB} \text{ since } FK \text{ and } AC \text{ are parallel, where } S(\triangle XYZ) \text{ denotes the area of } \triangle XYZ. \text{ Similarly, } \frac{S(\triangle AOB)}{S(\triangle ABC)} = \frac{BE}{BC} \text{ and } \frac{S(\triangle COB)}{S(\triangle ABC)} = \frac{CN}{CA}. \text{ From these, it is clear that } AF/AB + BE/BC + CN/CA = 1. \text{ So the correct answer is (e).}
\]

\[\text{Diagram}\]

7. Let \(x = 4\sqrt{(3 + \sqrt{3 + \sqrt{2}})}\). Then \(x^4 = 3 + \sqrt{3 + \sqrt{2}}\). Therefore \((x^4 - 3)^2 = 3 + \sqrt{2}\), i.e. \(x^8 - 6x^4 + 9 = 3 + \sqrt{2}\). Hence \((x^8 - 6x^4 + 6)^2 = 2\). The correct answer is therefore (e).

8. Note that if \(3 < x^3 \text{ and } x^5 < 6\), then \(3^5 < x^{15} < 6^3\). Hence we would have \(243 < 216\), a contradiction. So the system has no solution for \(n > 5\). For \(n = 4\), \(x = 1.45\) is a solution to the system. So the correct answer is (c).

9. The number of integers from 1000 to 1000000 is 999001. Among these 969 are squares of integers (namely \(32^2, 33^2, \ldots, 1000^2\)), 91 are cubes of integers (namely \(10^3, 11^3, \ldots, 100^3\)) and 7 are sixth powers of integers (namely: \(46^6, 56^6, \ldots, 106^6\)). Hence the number of integers which are neither squares nor cubes of integers is 999001 - 969 - 91 + 7 = 997948. So the correct answer is (d).

10. Let \(P(w)\), \(P(b)\) and \(P(r)\) be the probabilities that the ball drawn is white, black and red respectively. By symmetry, \(P(w) = P(b) = P(r)\). Also \(P(w) + P(b) + P(r) = 1\). From these we conclude that \(P(r) = 1/3\). Hence the correct answer is (b).
1. Let \( A_i, B_i, C_i \) \((i = 1, 2, 3)\) be the \(i\)th statements made by Grace, Helen and Mary respectively. The following combination of two statements made by each of the three ladies is consistent: \( A_2, A_3; B_1, B_2; \) and \( C_1, C_2. \) This combination leads to the conclusion that Grace is 23, Mary 22 and Helen 25 years old. All other combinations led to a contradiction. For example, the combination: \( A_1, A_2, B_1, B_2, \) and \( C_1, C_3 \) leads to the contradictory conclusions that Helen is both 24 and 25 years old.

2. We have: \((n!)^2 = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1\)

\[
= \prod_{t=0}^{n-1} (n-t) \times (t+1)
\]

\[
> \prod_{t=0}^{n-1} n
\]

since \((n-t)(t+1) = n + nt - t^2 - t = n + tn - (t-1)n > n\) for \(t > 0.\)

This proves that \((n!)^2 > n\), as required.

3. We first note that the equation has no solution in positive integers when \(x = 1\) or 2. Thus we may assume that \(x \geq 3.\) Clearly, \(y\) is not divisible by 2 and 3. Hence \(y = 6k \pm 1,\) where \(k > 1.\) The given equation becomes:

\[
3 \times 2 \times x^{x+1} = \left(6k \pm 1\right) = 36k^2 \pm 12k + 1,
\]

i.e.

\[
2^{x-2} = k(3k \pm 1).
\]

If \(k = 1,\) then we have \((x, y) = (3, 5), (4, 7).\)

If \(k \geq 2,\) then \(k(3k \pm 1)\) contains an odd factor while \(2^{x-2}\) does not. We thus conclude that the only solutions in positive integers are \((x, y) = (3, 5), (4, 7).\)

4. Let \(x = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_n}{n!}\)

Since \(a_k \geq 0\) for \(k = 2, 3, \ldots, n,\) we have

\[
a_2 /2! + a_3 /3! + \ldots + a_n /n! > 0.
\]

Thus \(x > a_1\) and so \([x] \geq a_1.\) Moreover,

\[
x = a_1 + \frac{a_2}{2!} + a_3 /3! + \ldots + a_n /n!
\]

\[
\leq a_1 + \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{(n-1)/n!}{(k-1)!}
\]

\[
= a_1 + \sum_{k=2}^{n} \frac{(k-1)}{k!}
\]

\[
= a_1 + \sum_{k=2}^{n} \left[1/(k-1)! - 1/k!\right]
\]

\[
= a_1 + 1 - 1/n!.
\]

Thus \([x] \leq a_1.\) Since we have shown that \([x] \geq a_1,\) we have \([x] = a_1.\)
Next, we have for fixed \( k \geq 2 \).

\[
k!x = k!(a_1 + a_2/2! + \ldots + a_k/k! + \ldots + a_n/n!)
= k!(a_1 + a_2/2! + \ldots + a_k/k!) + k!(a_{k+1}/(k+1)! + \ldots + a_n/n!)
\]

\[= I + II, \quad \text{where}
I = k!(a_1 + a_2/2! + \ldots + a_k/k!) \quad \text{and} \quad II = k!(a_{k+1}/(k+1)! + \ldots + a_n/n!).
\]

Obviously, \( I \) is an integer. For \( II \), we have

\[
i = k!\left(\sum_{i=k+1}^{n} \frac{a_i}{i!}\right) < k!\left\{\sum_{i=k+1}^{n} \frac{i-1}{i!}\right\}
= k!\left(\sum_{i=k+1}^{n} \frac{1}{(i-1)!} - \frac{1}{i!}\right) = k!\left(1/k! - 1/n!\right)
= 1 - k!/n! < 1.
\]

Hence \([k!x] = k!(a_1 + a_2/2! + \ldots + a_k/k!).\)

Similarly, \([k-1]!x] = (k-1)!(a_1 + \ldots + a_{k-1}/(k-1)!).\)

Thus \([k!x] - [(k-1)!x] k = k! a_k/k! = a_k.\)

5 The problem is equivalent to arranging the objects \( a_1, \ldots, a_{n-1}, b_1, \ldots, b_{m+1} \) while keeping the \( a \)'s and the \( b \)'s in their natural order. The answer is \((m+n)!/(n-1)!)(m+n)!\). However, those with \( b_m b_{m+1} \) occurring together but not at the end are repetitions. There are \((m+n-2)!/(n-2)! m!\) of these.

Therefore the answer is;

\[(m + n)! / (n - 1)! (m + 1)! - (m + n - 2)!/(n - 2)! m!.\]

6. We prove the identity by induction. Clearly the identity is true for \( n = 1 \).

Assume that it is true for \( n = k \). For \( n = k + 1 \), we have

\[
i_{1+1} \ldots i_{k+1} \frac{1}{a_1 + \ldots + a_{k+1} \sum_{k+1}^{k+1} a_i} \sum_{i=k+1}^{k+1} \frac{a_i + a_{i-2}}{a_1 + \ldots + a_{i-2}} = \frac{1}{a_1 + \ldots + a_{k+1}} \sum_{i=k+1}^{k+1} \frac{a_i}{a_1 + \ldots + a_{k+1}}
\]

Hence the identity is true for \( n = k + 1 \). This proves the identity for \( n \geq 1 \).

(ii) Given that \( N_1 = i_1, \ldots, N_{10} = i_{10} \), where \( i_1, \ldots, i_{10} \) is a permutation of \( 1, 2, \ldots, 10 \), the probability that the colours of the 10 balls drawn are all distinct is

\[
i_1 i_2 \ldots i_{10} / i_1 (i_1 + i_2) \ldots (i_1 + i_2 + \ldots + i_{10}).\]

But

\[P(N_1 = i_1, \ldots, N_{10} = i_{10}) = 1/10!\]. So the probability that the colours of the 10 balls drawn are all distinct is

\[
\sum_{i=1}^{10} \left[ i_1 i_2 \ldots i_{10} / i_1 (i_1 + i_2) \ldots (i_1 + \ldots + i_{10}) \right] [1/10!]
\]

\[= 1/10! \quad \text{by (i)}.\]