Attempt as many questions as you can. Circle your answers on the Answer Sheet provided.

Each question carries 5 marks

1. The largest positive integer which divides $n^5 - n$ for all positive integers $n$ is:
   (a) $1$, (b) $15$, (c) $30$, (d) $60$, (e) none of the preceding

2. Let $r, s$ be roots of $ax^2 + bx + c = 0$. Then the equation whose roots are $ar + b$ and $as + b$ is:
   (a) $x^2 - bx - ac = 0$
   (b) $x^2 - bx + ac = 0$
   (c) $x^2 + 3bx + ca + 2b^2 = 0$
   (d) $x^2 + 3bx - ca + 2b^2 = 0$
   (e) none of the preceding.

3. The remainder when $x^{120} + 1$ is divided by $(x + 1)^2$ is:
   (a) $2$, (b) $x + 3$, (c) $-x + 1$, (d) $2x + 4$, (e) none of the preceding

4. Let $m, n$ be integers with $1 < m \leq n$. We define:

   $$f(m, n) = (1 - \frac{1}{m}) (1 - \frac{1}{m+1}) (1 - \frac{1}{m+2}) \ldots (1 - \frac{1}{n}).$$

   Then $f(2, n) + f(3, n) + \ldots + f(n, n)$ is equal to:
   (a) $1/n$, (b) $n/2$, (c) $(n-1)/2$, (d) $n(n-1)$, (e) none of the preceding

5. A line segment is divided into three random parts. Then the probability that these three parts form the sides of a triangle is:
   (a) $1/4$, (b) $1/3$, (c) $1/2$, (d) $1/5$, (e) none of the preceding

6. Let $695 = a_1 + a_2.2! + a_3.3! + \ldots + a_n.n!$, where $a_1, a_2, \ldots, a_n$ are integers and $0 \leq a_k \leq k$ for each $k$. Then $a_4$ is equal to:
   (a) $0$, (b) $1$, (c) $2$, (d) $3$, (e) none of the preceding

7. Let $S$ be the solution set of the simultaneous equations $(x-1)^2 + (y+2)^2 + (z-5)^2 = 64$ and $(x+3)^2 + (y-1)^2 + (z+7)^2 = 25$, where $x, y$ and $z$ are real numbers. Then,
(a) S is an empty set.
(b) S is a singleton.
(c) S is a finite set with more than one element.
(d) S represents a straight line.
(e) none of the preceding.

8. The value of \( \int_{\pi/3}^{2\pi/3} \frac{x}{\sin x} \, dx \) is equal to:

(a) \(-\frac{\pi}{2} \ln 3\)
(b) \(-\frac{\pi}{3} \ln 3\)
(c) \(-\frac{\pi}{4} \ln 3\)
(d) \(-\frac{\pi}{6} \ln 3\)
(e) none of the preceding

9. Given that \(2x + y = 12\), the maximum value of \(\log_4 x + \log_2 y\) is:

(a) \(9/2\)
(b) 4
(c) 7/2
(d) 7/3
(e) none of the preceding

10. Let \(A\) be a 3 by 3 determinant such that all the entries of \(A\) are between \(-1\) and 1 inclusive of \(-1\) and 1. Then the maximum value of \(A\) is:

(a) 4
(b) 4.5
(c) 5
(d) 5.5
(e) none of the preceding.
SATURDAY, 29 JUNE 1985

INTERSCHOOL MATHEMATICAL COMPETITION 1985

ATTEMPT AS MANY QUESTIONS AS YOU CAN.

PART B

SATURDAY, 29 JUNE 1985

1100 – 1300

1. Let \( \triangle ABC \) be a triangle with \( \angle A = 4 \angle C \) and \( \angle B = 2 \angle C \). Show that \( \frac{BC + CA}{AB} = \frac{BC}{CA} \).

2. \( X, Y \) and \( Z \) are integers such that \( X^3 + 3Y^3 + 9Z^3 = 0 \). Prove that \( X = Y = Z = 0 \).

3. A real polynomial \( p(x) = ax^2 + bx + c \) (\( a > 0 \) and \( b > 0 \)) is such that \( |p(x)| \leq 1 \) for \( |x| \leq 1 \). Let \( q(x) = cx^2 + bx + a \). Show that \( |q(x)| \leq 2 \) for \( |x| \leq 1 \).

4. Prove that \( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \) for \( n > 1 \) is never an integer.

5. Denote by \( (x, y) \) the greatest common divisor of two positive integers \( x \) and \( y \). Let \( a \) and \( b \) be two positive integers such that \( (a, b) = 1 \) and let \( p \geq 3 \) be a prime. Denote \( aP + bP \).

6. Let \( n \) and \( r \) be integers with \( 0 < r \leq n \).

\( i \) Show that

\[ \binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \cdots + \binom{r}{r} = \binom{n+1}{r+1}. \]

Assume \( r \geq 1 \) and let \( \{A_1, A_2, \ldots, A_{\binom{n}{r}}\} \) be the collection of \( r \)-element subsets of the set \( S = \{1, 2, \ldots, n\} \). For each \( i = 1, 2, \ldots, \binom{n}{r} \), denote by \( m_i \) the smallest number in \( A_i \). Show that

\( ii \) \( 1 \leq m_i \leq n-r+1 \);

\( iii \) \( m_1 + m_2 + \ldots + m_{\binom{n}{r}} = \sum_{m=1}^{n-r+1} \binom{n-r}{r-1} m \); and

\( iv \) the arithmetic mean of \( m_1, m_2, \ldots, m_{\binom{n}{r}} \) is \( \frac{(n+1)}{(r+1)} \).
1. We have \( n^5 - n = n(n-1)(n+1)(n^2+1) \). As \( n-1, n \) and \( n+1 \) are three consecutive integers, we see that \( (n-1)n(n+1) \) is divisible by both 2 and 3 and hence by 6. Moreover if the unit digit of \( n \) is 0, 1, 4, 5, 6, 8 or 9, then \( (n-1)n(n+1) \) will also be divisible by 5 and hence by 30; while if the unit digit of \( n \) is 2, 3, or 7, then \( n^2+1 \) will be divisible by 5. Hence, in any case \( n^5 - n \) is always divisible by 30. As \( n^5 - n \) is equal to 30 when \( n = 2 \), the correct answer is (c).

2. As \( r, s \) are roots of \( ax^2 + bx + c = 0 \), we have \( r+s = -b/a \) and \( rs = c/a \). Now, \( (ar+b)+(as+b) = a(r+s)+2b = b \) and \( (ar+b)(as+b) = a^2rs+ab(r+s)+b^2 = ac-b^2+b^2 = c \). So the equation whose roots are \( ar+b \) and \( as+b \) is: \( x^2 - bx + ac = 0 \). Hence the correct answer is (b).

3. Let \( x^{120} + 1 = (x + 1)^2 Q(x) + ax + b \). Put \( x = -1 \), we have

\[-a + b = 2 \quad \text{(I)}\]

Differentiating the above equation with respect to \( x \), we have:

\[120x^{119} = 2(x+1)Q(x) + (x+1)^2 Q'(x) + a.\]

Put \( x = -1 \), we obtain:

\[a = -120 \quad \text{(II)}\]

From (I) and (II), we obtain \( b = -118 \). So the remainder is \(-120x - 118\). The correct answer is (e).

4. We have \( f(m, n) = [(m-1)m(m+1)\ldots(n-1)]/[m(m+1)(m+2)\ldots n] = (m-1)/n \). Thus \( f(2, n) + f(3, n) + \ldots + f(n, n) = [1 + 2 + \ldots + (n-1)]/n = (n-1)/2 \). Therefore the correct answer is (c).

5. Without loss of generality, we may assume that the line segment is the unit interval on the real line from 0 to 1. To divide the line segment into three random parts, we need only to choose two random numbers \( 0 \leq x \leq y \leq 1 \). All such pairs \((x, y)\) form a triangle with area 1/2 in the \(xy\)-plane as shaded in the following figure. The three parts will form the sides of a triangle if the inequalities as given below hold:
(i) \( x + (y-x) > 1 - y \) \( \Rightarrow \) \( y > 1/2 \) i.e. \( y > 1/2 \)

(ii) \( x + (1 - y) > y - x \) \( \Rightarrow \) \( y - x < 1/2 \) i.e. \( y - x < 1/2 \)

(iii) \( (y - x) + (1 - y) > x \) \( \Rightarrow \) \( x < 1/2 \) i.e. \( x < 1/2 \).

All pairs \((x, y)\) satisfying (i), (ii) and (iii) above form a triangle with area \(1/8\), as indicated in the doubly shaded region in the figure. Hence, the required probability is \([1/8] / [1/2] = 1/4\). The correct answer is (a).

6. The largest \( n \) with \( n! \leq 695 \) is 5. As \( 5 \times 5! = 600 < 695 \), we have \( a_5 = 5 \). Now \( 695 - 600 = 95 \) and the largest \( n \) with \( n! < 95 \) is 4. As \( 3 \times 4! = 72 < 95 \), we have \( a_4 = 3 \). So the correct answer is (d).

7. In \( \mathbb{R}^3 \), the equation \((x-1)^2 + (y+2)^2 + (z-5)^2 = 64\) represents a sphere with centre at the point \( A(1, -2, 5) \) and radius equal to 8 units. The equation \((x+3)^2 + (y-1)^2 + (z+7)^2 = 25\) also represents a sphere with centre at the point \( B(-3, 1, -7) \) and radius equal to 5 units. Now, \( AB = 13 = 8 + 5 \), from which we conclude that \( S \) must be a singleton. Hence the correct answer is (b).

8. Let \( y = \pi - x \). Then we have

\[
\int_{\pi/3}^{2\pi/3} \frac{2\pi}{3} \frac{(x/\sin x)}{dx} = - \int_{2\pi/3}^{\pi/3} (\pi - y)/\sin (\pi - y) \ dy
\]

\[
= \int_{\pi/3}^{2\pi/3} (\pi - y) \sin y \ dy
\]

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\[ \int_{\pi/3}^{2\pi/3} \frac{\tan y}{\sin y} \, dy - \int_{\pi/3}^{2\pi/3} \frac{\tan y}{\sin y} \, dy = \ln(\csc y - \cot y) \bigg|_{\pi/3}^{2\pi/3} \]

\[ \int_{\pi/3}^{2\pi/3} \frac{\tan y}{\sin y} \, dy. \]

Hence, \[ \int_{\pi/3}^{2\pi/3} \frac{x}{\sin x} \, dx = \pi/2 \ln(\csc y - \cot y) \bigg|_{\pi/3}^{2\pi/3} \]

\[ = \frac{\pi}{2} \ln 3. \]

So the correct answer is (a).

9. We have \( y = 12 - 2x = 2(6 - x), \) \( y > 0 \) and \( x > 0. \) Hence \( 0 < x < 6. \)

Moreover, \( \log_4 x + \log_2 y = \log_2 x / \log_2 4 + \log_2 2 (6-x) \)
\[ = (1/2) \log_2 x + 1 + \log_2 (6 - x) \]
\[ = (1/2)(2 + \log_2 x + 2\log_2 (6 - x)) \]
\[ = (1/2)(2 + \log_2 x (6 - x)^2). \]

Let \( f(x) = x(6 - x)^2. \) Then \( f'(x) = 3(x - 2)(x - 3) \) and \( f''(x) = 6(x - 4). \) We see that \( f'(x) = 0 \) iff \( x = 2 \) for \( 0 < x < 6. \) Also, \( f''(2) = -12 < 0. \) So \( f(x) \) has a maximum value when \( x = 2. \) Also when \( x = 2, \) \( \log_4 x + \log_2 y = (1/2)(2 + \log_2 32) = 7/2. \)
The correct answer is therefore (c).

10. Let \( (aij) \) be the \( 3 \times 3 \) matrices with entries \( a_{ij} \) and let \( A_{ij} \) be the determinant by deleting the \( i \)th row and the \( j \)th column from \( A. \) Then
\[ A = \sum_{i=1}^{3} (-1)^i \cdot a_{ij}A_{ij} \text{ for } i = 1, 2, 3. \]

It is easy to see that we can change \( |aij| \) to 1 without decreasing the value of \( A. \) Hence \( A \) can achieve its maximum value when \( |aij| = 1 \) for all \( i, j. \) Thus we may assume that \( |aij| = 1 \) for all \( i, j. \)
Next, we note that \( A \) is the volume of the parallelepiped formed by the vectors \( (a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}) \) and \( (a_{31}, a_{32}, a_{33}). \) Hence \( A \leq \)
Finally, we see that $A$ is the sum of six terms which are either $+1$ or $-1$. Since $A < 6$, at least one of these six terms must be $-1$. Hence $A \leq 4$. As

\[
\begin{vmatrix}
1 & -1 & 1 \\
1 & 1 & -1 \\
1 & 1 & 1 \\
\end{vmatrix} = 4
\]

we conclude that the maximum value of $A$ is 4. The correct answer is therefore (a).
1. We have $\angle A = 4\angle C$, $\angle B = 2\angle C$ and $\angle A + \angle B + \angle C = \pi$. Hence $\angle C = \pi/7$, $\angle B = 2\angle C$ and $\angle A = 4\angle C$. From this, we have: $BC = k\sin(4\angle C)$, $CA = k\sin(2\angle C)$ and $AB = k\sin(\angle C)$ for some constant $k$. Therefore we need only to show now that $(\sin(4\angle C) + \sin(2\angle C))\sin(\angle C) = \sin(4\angle C)\sin(2\angle C)$. Indeed, we have:

$$(\sin(4\angle C) + \sin(2\angle C))\sin(\angle C) = (2\sin(3\angle C)\cos(\angle C))\sin(\angle C) = \sin(3\angle C)\sin(2\angle C) = \sin(\pi - 3\angle C)\sin(2\angle C) = \sin(4\angle C)\sin(2\angle C),$$

as required.

2. Suppose to the contrary that there exist integers $X$, $Y$, and $Z$ not all zero satisfying the given equation. Without loss of generality, we may assume the greatest common divisor of $X$, $Y$, and $Z$ is 1. As $X^3 + 3Y^3 + 9Z^3 = 0$, we see that 3 divides $X$, say $X = 3A$ for some integer $A$. By substitution, we get $27A^3 + 3Y^3 + 9Z^3 = 0$. Dividing this equation throughout by 3, we get $Y^3 + 3Z^3 + 9A^3 = 0$. Therefore, 3 divides $Y$, say $Y = 3B$. Again, by substitution, we have $27B^3 + 3Z^3 + 9A^3 = 0$, i.e. $Z^3 + 3A^3 + 9B^3 = 0$. Hence, we have 3 divides $Z$. This however contradicts the fact that $X$, $Y$, and $Z$ have greatest common divisor 1. We thus conclude that $X = Y = Z = 0$, as required.

3. Substituting $x = 1$, 0 and $-1$ respectively into $p(x)$, we get

$$|a + b + c| \leq 1 \quad \text{and} \quad |c| \leq 1 \quad \text{and} \quad |a - b + c| \leq 1.$$ 

Therefore, $|a + b| = |a + b + c - c| \leq |(a + b + c)| + |c| \leq 2$ and $|a - b| = |a - b + c - c| \leq |a - b + c| + |c| \leq 2$. Now let $x$ be any real number with $|x| \leq 1$. If $c \geq 0$ then we have: $0 \leq cx^2 \leq c$ and $-b \leq bx \leq b$. Thus, $-2 \leq a - b = 0 + (-b) + a \leq q(x) = cx^2 + bx + a \leq c + b + a \leq 1$. Hence $|q(x)| \leq 2$ for $|x| \leq 1$, as required. On the other hand, if $c < 0$, we have $c \leq cx^2 \leq 0$ and $-b \leq bx \leq b$. Then, $-1 \leq c - b + a \leq q(x) = cx^2 + bx + a \leq 0 + b + a \leq 2$. This again gives $|q(x)| \leq 2$, which completes the proof.

4. Assume $\sum_{i=1}^{n} 1/i \in \mathbb{Z}$. We may assume that $n > 2$. Let the least common multiple of $1, 2, \ldots, n$ be $a$. There is an $r$ in $\mathbb{Z}$ ($r > 0$) with $2^r \leq n < 2^{r+1}$. Let $a = 2^r3^s5^t \ldots$ be the prime decomposition of $a$. Since $n > 2$, we have $a/2 \in \mathbb{Z}$. We
then have: \((\frac{1}{2}a)/2' \notin \mathbb{Z}\) by definition of \(r\), and

\[
(\frac{1}{2}a)/j \in \mathbb{Z} \text{ for } 1 \leq j \leq n, j \neq 2' \tag{*}
\]

But \(a/2 \in \mathbb{Z}\), and by assumption \(\frac{a}{j} \in \mathbb{Z}\) for \(1 < i < n, i \neq 1\), thus \(\frac{a}{j} \in \mathbb{Z}\). On the other hand, by (*) we find that \(\frac{a}{j} \notin \mathbb{Z}\). This is a contradiction.

5. We have \(a + b = \alpha t\) and \((a^t + b^t)/(a + b) = \alpha s\) for some positive integers \(t\) and \(s\). Therefore \(\alpha^2 ts = a^t + b^t = a^t + (\alpha t - a)^t = \alpha^t t^p - p\alpha t \alpha^t - 1 t^p - 1 + \ldots + p\alpha \alpha^t - 1 t^p - 1\). Hence \(\alpha s = \alpha^t t^p - 1 - p\alpha t^p - 2 t^p - 2 + \ldots + p\alpha p - 1\). From this, we see that \(\alpha \mid pa^t - 1\). We shall show that \((\alpha, a) = 1\). Suppose to the contrary that \((\alpha, a) = k > 1\). Let \(q\) be a prime factor of \(k\). Clearly, \(q\mid k\), \(q\mid \alpha\) and \(q\mid a\).

Since \(\alpha \mid a + b\), \(q\mid a + b\) and so \(q\mid b\). But then \((a, b) \neq 1\), a contradiction. Thus \((\alpha, a) = 1\), as required.

Now, since \(\alpha \mid pa^t - 1\) and \((\alpha, a) = 1\), it follows that \(\alpha \parallel p\) and so \(\alpha = 1\) or \(\alpha = p\), which completes the proof.

6. (i) The identity can be proved by induction on \(n\), using the following result:

\[
\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}
\]

(ii) Let \(A_r = \{a_1, a_2, \ldots, a_r\}\) where \(m_1 = a_1 < a_2 < \ldots < a_r\). Then \(m_1 + m_2 + \ldots + m_{r-1} + 1 \geq \ldots \geq a_1 + (r-1) = m_1 + (r-1)\), which implies that \(m_i \leq n-r+1\). The fact that \(1 \leq m_i\) is trivial.

(iii) For each \(m\) with \(1 \leq m \leq n - r + 1\), the number of \(r\)-element subsets of \(S\) containing \(m\) as the smallest number is \(\binom{n-m}{r-1}\). Thus

\[
m_1 + m_2 + \ldots + m_n = \sum_{m=1}^{n-r+1} m \left( \binom{n-m}{r-1} \right)
\]

(iv) By applying the identity in (i) repeatedly, we have

\[
m_1 + m_2 + \ldots + m_n = \sum_{m=1}^{n-r+1} m \left( \binom{n-m}{r-1} \right)
\]
Thus the arithmetic mean of $m_1, m_2, \ldots, m_n$ is

$$\frac{n+1}{r+1} \cdot \binom{n}{r} = (n+1)/(r+1).$$