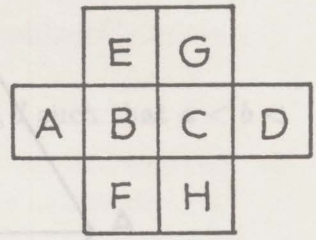
Solutions to Part A

1.

$$x = \frac{\log_2 3}{\log_2 2}\frac{\log_2 4}{\log_2 3}\cdots\frac{\log_2 8}{\log_2 7} = \frac{\log_2 8}{\log_2 2} = 3.$$

2. Label the chessboard as shown. Then B has 6 touching squares, so B must have 1 or 8. Similarly for C. Thus, either $B = 1$ and $C = 8$, or $B = 8$ and $C = 1$. Also, $B = 1$ implies $D = 2$. Similarly for A. Therefore, B determines A, C, and D. The numbers left are 3, 4, 5, 6. E and F touch A, so two of these numbers cannot be in E or F. The only choice left is between G and H. This choice determines E, F, G and H. Since there are two choices for B, and for each of these, there are two choices for G, there are altogether four possible assignments.



3. $x^{256} - 256^{32} = x^{256} - 2^{256} = 0$ has only two real roots: 2 and -2, so $2^2 + (-2)^2 = 8$.

4.

$$\sum_{n=0}^{\infty}\frac{2n}{2^n} = 2\sum_{n=1}^{\infty}\frac{n}{2^n} = 2\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{1}{2^n} = 2\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}\frac{1}{2^n} = 2\sum_{i=1}^{\infty}\frac{1}{2^i}\sum_{n=i}^{\infty}\frac{1}{2^{n-i}}$$

$$= 2\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{1}{1-\frac{1}{2}} = 4\sum_{i=1}^{\infty}\frac{1}{2^i} = 4.$$

5. $x = \sqrt{5+\sqrt{3+x}} \Rightarrow x^2 - 5 = \sqrt{3+x} \Rightarrow x^4 - 10x^2 - x + 22 = 0$ $\Rightarrow (x+2)(x^3 - 2x^2 - 6x + 11) = 0$. Let $f(x) = x^3 - 2x^2 - 6x + 11$. Then, $f(x) < 0$ for large negative x, $f(0) > 0$, $f(2) < 0$, $f(3) > 0$; these imply there are 3 real roots, two of which are less than 2. But $x > \sqrt{5} > 2$, so $2 < x < 3$.

6.

$$\left(\frac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}}\right)^2 = 2, \quad \text{and} \quad 3-2\sqrt{2} = (\sqrt{2}-1)^2,$$

so we have $\sqrt{2}-(\sqrt{2}-1)=1$.

7. Let b be the length of AC and c the length of AB. Then $b+c = 8+\sqrt{3}-\mathrm{BC} = 8+\sqrt{3}-4 = 4+\sqrt{3}$ and $b^2+c^2 = BC^2 = 16$. Since $(b+c)^2 = 19+8\sqrt{3}$, we get $2bc = 3+8\sqrt{3}$. The area is $\frac{1}{2}bc = \frac{1}{4}(3+8\sqrt{3})$.

8. Let O be the centre, $\angle AOB = x^\circ$, and $\angle AOD = y^\circ$. Then $3x+y = 360$ and $40+\frac{1}{2}y = \frac{1}{2}x$. Solving, we get $x = 110$, $y = 30$, so $\angle ACD = 15^\circ$.

9.

$$\frac{\binom{7}{2}\frac{10.8}{2!}}{\binom{14}{6}} = \frac{40}{143}.$$

10. Let W, L, D represent respectively Amy winning, losing, and drawing a game. The sequences that give Amy a higher total are permutations of (W,D,D,D), (W,W,L,D), (W,W,D,D), (W,W,W,L), (W,W,W,D) and (W,W,W,W). The total probability is

$$\binom{4}{1}\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)^3 + \binom{4}{2}\binom{2}{1}\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) + \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{6}\right)^2$$

$$+\binom{4}{1}\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right) + \binom{4}{1}\left(\frac{1}{2}\right)^3\left(\frac{1}{6}\right) + \left(\frac{1}{2}\right)^4 = \frac{229}{432}.$$

Solutions to Part B

1. Suppose the claim is true.

$$\frac{1}{a}+\frac{1}{d}=\frac{1}{b}+\frac{1}{c} \Rightarrow \frac{1}{a}-\frac{1}{b}=\frac{1}{c}-\frac{1}{d} \Rightarrow cd(b-a)=ab(d-c),$$

so b divides $cd(b-a)$. Since b is a prime, either b divides cd or b divides $b-a$. But b, c, and d are unequal primes, so b does not divide cd; also, $b \neq a$, so b does not divide $b-a$. These contradict the previous statement, so the claim is false.

2. Suppose $\frac{p}{q}$ is a rational root, where p, q are integers and $\gcd(p,q)=1$. Then

$$a_0 q^n + a_1 q^{n-1} p + \cdots + a_n p^n = 0. \qquad (1)$$

Hence p divides a_0 and q divides a_n, so both p and q are odd. It follows that $a_k q^{n-k} p^k$ is odd if and only if a_k is odd. LHS of (1) is even, so $a_0+a_1+\cdots+a_n$ is even. This contradicts the fact that $f(1)$ is odd.

3. There are 8.7.6 ways to sit in a circle. The first person can sit anywhere. The other 3 permute themselves clockwise in 6 ways, with one chair between every two persons. The extra chair can go in 4 possible positions, so the total number of ways is $4.6=24$. The probability is $\frac{24}{8.7.6}=\frac{1}{14}$.

4. We have $(\cos A+\cos B+\cos C)^2=0$, so

$$\cos^2 A+\cos^2 B+\cos^2 C + 2(\cos A\cos B+\cos B\cos C+\cos C\cos A)=0. \qquad (2)$$

However,

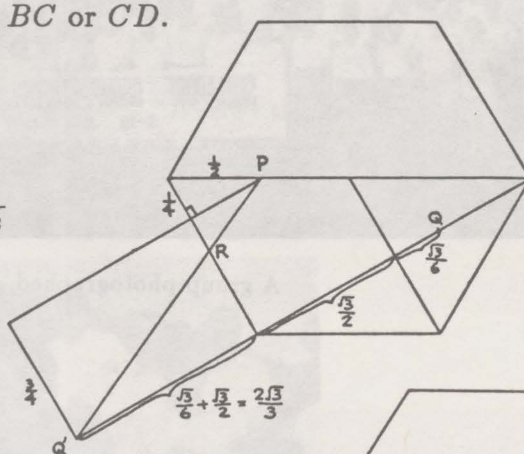
$$\begin{aligned} &2(\cos A\cos B+\cos B\cos C+\cos C\cos A)\\ &=\cos(A+B)+\cos(A-B)+\cos(B+C)\\ &\quad+\cos(B-C)+\cos(C+A)+\cos(C-A). \end{aligned} \qquad (3)$$

Let $A+B+C=D$. Then

$$\begin{aligned} &\cos(A+B)+\cos(B+C)+\cos(C+A)\\ &=\cos(D-C)+\cos(D-A)+\cos(D-B)\\ &=\cos D(\cos C+\cos A+\cos B)+\sin D(\sin C+\sin A+\sin B)=0. \end{aligned}$$

Moreover, $(\sin A + \sin B)^2 = (-\sin C)^2 = \sin^2 C$ and $(\cos A + \cos B)^2 = (-\cos C)^2 = \cos^2 C$, from which we get $2+2\cos(A-B) = 1$, i.e., $\cos(A-B) = -\frac{1}{2}$. Similarly, $\cos(B-C) = \cos(C-A) = -\frac{1}{2}$. Combining (2), (3) and these, we get $\cos^2 A + \cos^2 B + \cos^2 C = \frac{3}{2}$.

5. It is not difficult to see that the position of R which yields minimum $PR + QR$ must be on AB, BC or CD.

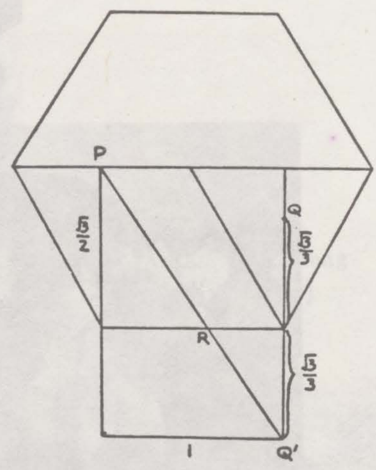


<u>R on AB.</u>

$PR + RQ$

$= \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{\sqrt{3}}{4} + \frac{2\sqrt{3}}{3}\right)^2}$

$= \sqrt{\frac{37}{12}}$

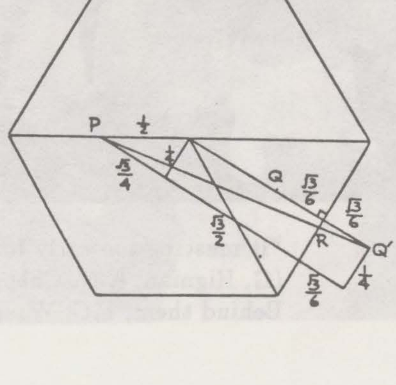


<u>R on BC.</u>

$PR + RQ$

$= \sqrt{1^2 + \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}\right)^2}$

$= \sqrt{\frac{37}{12}}.$



<u>R on CD.</u>

$PR + QR$

$= \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{6}\right)^2}$

$= \sqrt{\frac{31}{12}}.$

The minimum value is $\sqrt{\frac{31}{12}}$.