# An overview of classical integration theory

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## §1. Introduction

The theory of integration has a long history which dated back to two thousand years ago. However the modern theory began with I. Newton (1642-1727) and G. W. von Leibniz (1646-1716) in the seventeen century. The idea of fluxions, as Newton called his calculus, was developed further with applications to mechanics, physics and other areas.

The foundation of the modern theory of integration or what we now call classical integration theory, was laid by G. F. B. Riemann (1826-1866) in the nineteen century. This is also the integration theory which is taught in the undergraduate years at the university. However, in 1902, H. Lebesgue (1875-1941) following the work of others established what is now known as the Lebesgue integral, or in its abstract version measure theory. Of course, many great mathematicians, who came before and after Lebesgue, helped to initiate, to develop, and later perfected the theory. This is the theory that dominates the mathematics arena nowadays. It finds applications in virtually every branch of mathematical analysis.

However the Lebesgue integral has its defects. For example, it does not integrate the derivatives as the Newton calculus does. An integral that includes Lebesgue and is able to integrate the derivatives was first defined by A. Denjoy in 1912 and later another version by O. Perron in 1914. It was until 1921 that the two integrals were proved to be equivalent. The fact that it took so many years shows the difficulty of the proof at the time. There has been active research on the Denjoy-Perron integral since then until 1935 before the second world war.

Both the Denjoy and the Perron integrals were difficult to handle. The break-through came in 1957/58 when Henstock and Kurzweil gave independently a Riemann-type definition to the Denjoy-Perron integral. Not only that the definition is now easier, but also the proofs using the Henstock-Kurzweil integral are often simpler. Recently, there has been and still is active research on the subject, in particular, the extension to the n-dimensional spaces and applications to ordinary differential equations and trigonometric series.

In what follows, we describe briefly three typical approaches to the classical integration theory through the historical development of the subject in three stages, namely the original Riemann integral (section 2), Lebesgue theory (section 3), and Henstock theory (section 4). In the final section (section 5), we mention some recent advances of Henstock theory. We assume the reader has some familiarity with the Riemann integral.

## §2. Integration of continuous functions

The Riemann integral is well-known. It integrates continuous functions as well as some discontinuous functions. The definition appears in every calculus book. For completeness, we state it as follows. Let a function f be defined on $[a, b]$. Consider a division $D$ of $[a, b]$ given by

$$a = x_0 < x_1 < \ldots < x_n = b$$

with $\xi_1, \xi_2, \ldots, \xi_n$ so that $\xi_i$ is related to $[x_{i-1}, x_i]$ in some way for each $i$. Then a Riemann sum can be formed written

$$s(f, D) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

For convenience, $D$ always means the division given above and $s(f, D)$ the corresponding Riemann sum throughout this paper. A function $f$ is said to be Riemann integrable on $[a, b]$ if there is a real number A for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every division $D$ of $[a, b]$ satisfying $\xi_i - \delta < x_{i-1} \leq \xi_i \leq x_i < \xi_i + \delta$ for $i = 1, 2, \ldots, n$ we have

$$|s(f, D) - A| < \varepsilon.$$

Also, we write

$$A = \int_a^b f(x)dx,$$

and say that $A$ is the integral of $f$ on $[a, b]$. It is a standard exercise to show that every continuous function defined on $[a, b]$ is Riemann integrable there. In fact, Riemann proved only that if $f$ is uniformly continuous on

$[a, b]$ then $f$ is integrable in his sense there. But Bolzano stated a theorem which implies that every function continuous on a closed bounded interval is uniformly continuous, and Weierstrass proved it later in the 1870s. In short, they collectively proved that every continuous function is Riemann integrable.

Suppose we define

$$F(x) = \int_a^x f(t)dt \quad \text{for} \quad a \leq x \leq b.$$

We shall call $F$ the primitive of $f$ and shall use this term throughout the paper. If $f$ is continuous on $[a, b]$ then we can prove that the derivative $F'(x)$ exists and $F'(x) = f(x)$ for $x \in [a, b]$. This property describes the Newton integral. More precisely, a function $f$ is said to be Newton integrable on $[a, b]$ if there is a differentiable function $F$ such that $F'(x) = f(x)$ for every $x \in [a, b]$. The integral of $f$ on $[a, b]$ is $F(b) - F(a)$. In particular, every continuous function defined on $[a, b]$ is Newton integrable there.

The above two definitions, namely Riemann and Newton, are the classical definitions of the integral of continuous functions. The first is known as the constructive definition and the second as the descriptive definition.

Now we consider a third definition. The approach resembles that of using rational numbers to define an irrational number by means of what is known as Dedekind cut. Roughly, an irrational number $x$, say $x = \sqrt{2}$, is defined as the cut or the Dedekind cut between two sets of rational numbers $\{r; r^2 < 2\}$ and $\{r; r^2 > 2\}$. In the case of integration, we define some elementary functions, for example, step functions which are easy to integrate. Then we use elementary functions to define, like the Dedekind cut, the integral of more general functions, for example, continuous functions.

Let $f$ be a continuous function on $[a, b]$. Therefore $f$ is bounded on $[a, b]$. Consider a division $D$ of $[a, b]$. Define an upper step function $h$ and a lower step function $g$ obtained from $f$ and $D$ as follows :

$$h(x) = M_i \quad \text{and} \quad g(x) = m_i$$

when $x_{i-1} \leq x < x_i$ for $i = 1, 2, \ldots, n-1$, and when $x_{i-1} \leq x \leq x_i$ for $i = n$, where

$$M_i = sup\{f(t); x_{i-1} \leq t \leq x_i\},$$
$$m_i = inf\{f(t); x_{i-1} \leq t \leq x_i\}.$$

Obviously, $g(x) \leq f(x) \leq h(x)$ for all $x \epsilon [a, b]$, and we have

$$\sup_D \int_a^b g(x)dx \leq \inf_D \int_a^b h(x)dx.$$

Here the supremum is taken over all $D$ and the corresponding lower step functions $g$ and the infimum over all $D$ and the corresponding upper step functions.

If the above two values are equal, then the value is defined to be the integral of $f$ on $[a, b]$. Again, it is well-known that the above definition is equivalent to the Riemann integral. By the equivalence of two integrals, we mean : if a function is integrable in one sense, then it is also integrable in another sense, and vice versa.

We may present the above definition in a different way. Let $g$ and $h$ be respectively the lower and the upper step functions of $f$ and put

$$G(x) = \int_a^x g(t)dt \quad \text{and} \quad H(x) = \int_a^x h(t)dt \quad \text{for} \quad a \leq x \leq b.$$

Since $g$ and $h$ are step functions, therefore $G$ and $H$ are continuous piecewise linear functions. Furthermore,

$$G'(x) \leq f(x) \leq H'(x)$$

for all except a finite number of $x$ in $[a, b]$. In order to avoid exceptional points, we may rephrase the inequalities as follows :

$$\bar{D}G(x) \leq f(x) \leq \underline{D}H(x) \quad \text{for all } x.$$

Here we allow $\underline{D}H(x)$ to take the value $+\infty$ but not $-\infty$. Similarly, $\bar{D}G(x)$ is allowed to take the value $-\infty$ but not $+\infty$. As usual, $\bar{D}G(x)$ denotes the upper derivative defined by

$$\bar{D}G(x) = \limsup_{h \to 0} \frac{G(x+h) - G(x)}{h},$$

and $\underline{D}H(x)$ the lower derivative defined by

$$\underline{D}H(x) = \liminf_{h \to 0} \frac{H(x+h) - H(x)}{h}.$$

For convenience, if the above inequalities hold $H$ is called a major function of $f$ and $G$ a minor function of $f$ on $[a, b]$.

**Theorem 1:** *A function $f$ is Riemann integrable on $[a, b]$ if and only if $f$ has the major and the minor functions which are piecewise linear and continuous and*

$$inf\{H(b) - H(a)\} = sup\{G(b) - G(a)\}$$

*where the infimum is over all the above major functions $H$ of $f$ on $[a, b]$ and the supremum over all the above minor functions $G$ of $f$ on $[a, b]$.*

So far we have given three different approaches to integration of continuous functions, namely, constructive, descriptive and one that uses major and minor functions.

## §3. Absolute integrals

The Riemann integral is absolute in the sense that if $f$ is integrable on $[a, b]$, so is $|f|$. However the Newton integral is nonabsolute in the sense that if $f$ is integrable, we cannot say that $|f|$ is. The Lebesgue integral is an extension of Riemann, and it is again an absolute integral.

To proceed, we take a monotone sequence of continuous functions, say, $\{f_n\}$ with $f_1(x) \leq f_2(x) \leq \ldots$ and $f_n(x) \to f(x)$ and $n \to \infty$ for all $x \in [a, b]$. In the theory of Riemann, it does not matter if we ignore a set of finite points since a function has no area below a point. Similarly, in the theory of Lebesgue, it does not matter if we ignore a set of measure zero. More precisely, a set $X$ is said to be of measure zero if for every $\varepsilon > 0$ there is a countable number of open intervals $I_1, I_2, \ldots$ such that

$$\bigcup_{i=1}^{\infty} I_i \supset X \quad \text{and} \quad \sum_{i=1}^{\infty} |I_i| < \varepsilon$$

where $|I_i|$ denotes the length of the interval $I_i$. Hence we may consider a nondecreasing sequence of continuous functions $\{f_n\}$ with

$$f_1(x) \leq f_2(x) \leq \ldots \quad \text{and} \quad \lim_{n \to \infty} f_n(x) = f(x)$$

almost everywhere in $[a, b]$ i.e. everywhere except perhaps a set $X$ of measure zero in $[a, b]$. Suppose the integrals of $f_n$ form a bounded nondecreasing sequence. That is,

$$\int_a^b f_1(x)dx \leq \int_a^b f_2(x)dx \leq \ldots \quad \text{and} \quad sup\{\int_a^b f_n(x)dx;\, n \geq 1\} < +\infty.$$

Since every bounded nondecreasing sequences has a limit, therefore we may define

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Similarly, we may consider a nonincreasing sequence of continuous functions $\{f_n\}$ converging almost everywhere to a function $f$ and define

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx,$$

if the limit exists. We apply once more the monotone convergence (nondecreasing or nonincreasing) as before with continuous functions replaced by the limits (almost everywhere) of monotone convergent sequence of continuous functions. What we obtain by taking monotone convergence twice is the family of all Lebesgue integrable functions if the integrals so defined are finite. In other words, the family of all Lebesgue integrable functions on $[a, b]$ is the smallest family containing continuous functions and closed under monotone convergence in the above-mentioned sense. It is interesting to note that if we regard two functions as equal when they are equal pointwise almost everywhere, then we do not obtain any more new elements by taking monotone convergence more than twice.

There are at least a few dozen different ways of presenting the Lebesgue integral. The above approach is classical. In what follows, we shall give respectively descriptive definition, constructive definition, and one that uses major and minor functions.

In the case of continuous functions $f$, we know that the primitive $F$ of $f$ is continuous and indeed differentiable everywhere. This is no longer true for Lebesgue integrable functions. The most we can say is that $F$ is absolutely continuous on $[a, b]$ and differentiable almost everywhere. More precisely, a function $F$ is said to absolutely continuous on $[a, b]$ if for every $\varepsilon > 0$ there is $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$ satisfying

$$\sum_i |b_i - a_i| < \eta \quad \text{we have} \quad \sum_i |F(b_i) - F(a_i)| < \varepsilon.$$

Intuitively, a continuous function maps a small interval in the domain into another small interval in the range. An absolutely continuous function maps a collection of small intervals (which are obtained by cutting up a small $\eta$ interval) in the domain into another collection of small intervals in the range. Note that if we put the collection of small intervals in the range together we obtain a small interval of length less than $\varepsilon$. It is easy to see that an absolutely continuous function is continuous, but not conversely.

In fact, the absolute continuity and almost everywhere differentiability of the primitive function $F$ characterize completely the Lebesgue integral. And we have

**Theorem 2:** *A function $f$ is Lebesgue integrable on $[a,b]$ if and only if there is an absolutely continuous function $F$ such that $F'(x) = f(x)$ almost everywhere in $[a,b]$.*

The above theorem provides a descriptive definition to the Lebesgue integral.

The Riemann-type definition to the Lebesgue integral came much later. It was given by E. J. McShane (1904-1989). To motivate, let us attempt to prove again that a continuous function is integrable in some sense without reference to uniform continuity. First, $f$ is continuous on $[a,b]$, for every $x \in [a,b]$ there is $\delta(x) > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad y \in (x - \delta(x), x + \delta(x)).$$

Here $\delta(x)$ is no longer a constant and it depends on $x$. However it is still possible to form a Riemann sum. Consider the family of all open intervals $(x-\delta(x), x+\delta(x))$ for $x \in [a,b]$. Using the bisection method, we can prove that there exists a division

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

with $\xi_1, \xi_2, \ldots, \xi_n$ such that $[x_{i-1}, x_i] \subset (\xi_i - \delta(x_i), \xi_i + \delta(x_i))$ for $i = 1, 2, \ldots, n$. Therefore a Riemann sum can be formed using $x_i$'s and $\xi_i$'s.

As usual, in order to prove the existence of a limit without knowing what the limit is, we apply the Cauchy criterion. Given $\varepsilon > 0$, we want to show that under certain conditions for any two Riemann sums $s(f, D_1)$ and $s(f, D_2)$ we have

$$|s(f, D_1) - s(f, D_2)| < \varepsilon,$$

where

$$s(f, D_1) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),$$

$$s(f, D_2) = \sum_{j=1}^{m} f(\eta_j)(y_j - y_{j-1}).$$

Take $D_3 \subset D_1 \cap D_2$, by which we mean $D_3$ is given by

$$a = t_0 < t_1 < \ldots < t_p = b$$

and $[t_{k-1}, t_k]$ lies in $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ for some $i$ and some $j$. Note that the new Riemann sum formed using $D_3$ as a finer division of $D_1$ may have the associated points $\xi_i$ which lie outside $[t_{k-1}, t_k]$. As a consequence, in order that the above proof goes through we cannot require the associated points to lie inside the corresponding intervals. Hence we arrive at the following definition. A function $f$ defined on $[a, b]$ is said to McShane integrable if there is a real number $A$ for every $\varepsilon > 0$ there is $\delta(\xi) > 0$ such that for any division $D$ given by

$$a = x_0 < x_1 < \ldots < x_n = b$$

with $\xi_1, \ldots, \xi_n$ and satisfying $[x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i$ we have

$$\left|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A\right| < \varepsilon.$$

**Theorem 3:** *A function $f$ is Lebesgue integrable on $[a, b]$ if and only if it is McShane integrable there.*

The above provides a Riemann-type or constructive definition to the Lebesgue integral.

The Lebesgue integral can also be defined as a Dedekind cut using continuous functions as the elementary functions in place of step functions as in the Riemann integral. Here we have to define a different set of major and minor functions.

A function $F$ is said to be of bounded variation on $[a, b]$ if there is a positive number $M$ such that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \le M$$

for all divisions $a = x_0 < x_1 < \ldots < x_n = b$. Graphically, we add up all the ups and downs of a given function over $[a, b]$ and if the total is finite then it is a function of bounded variation.

**Theorem 4:** *A function $f$ is Lebesgue integrable on $[a, b]$ if and only if $f$ has the major and minor functions which are of bounded variation and*

$$inf\{H(b) - H(a)\} = sup\{G(b) - G(a)\}$$

*where the infimum is over all the above major functions $H$ of $f$ on $[a, b]$ and the supremum over all the minor functions $G$ of $f$ on $[a, b]$.*

The theory of the Lebesgue integral has been made abstract and is known as measure theory. It is a general belief that many theorems hold true in the Lebesgue theory because of the so-called countable additivity property. Henstock showed otherwise that finite additivity is enough to generate all the desired theorems and more. We shall define the Henstock integral in the next section.

## §4. Nonabsolute integrals

As mentioned in the previous section, the Lebesgue integral is absolute. Hence it cannot integrate all derivatives. For example, let $F(x) = x^{-2} sin\ x^{-2}$ when $x \neq 0$ and $F(0) = 0$. Then its derivative $F'$ exists everywhere and is Newton integrable on $[0, 1]$ but not Lebesgue integrable there. There have been many attempts to define integrals which include both the Lebesgue and Newton integrals. They include Denjoy, Perron, Kurzweil and Henstock among others.

To motivate, consider a differentiable function $F$ such that $F'(x) = f(x)$ for $x \in [a, b]$. Then at each $\xi$ and for every $\varepsilon > 0$ there is $\delta(\xi) > 0$ such that whenever $\xi - \delta(\xi) < u \leq \xi \leq v < \xi + \delta(\xi)$ we have

$$|F(v) - F(u) - f(\xi)(v - u)| \leq \varepsilon|v - u|.$$

Now consider a division $D$ of $[a, b]$ given by

$$a = x_0 < x_1 < \ldots < x_n = b$$

with $\xi_1, \xi_2, \ldots, \xi_n$ such that $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, \ldots, n$. Such division exists as mentioned before the definition of the McShane integral and is said to be $\delta$-fine. Any Riemann sum using the above division gives the following

$$|F(b) - F(a) - \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})| \leq \varepsilon|b - a|.$$

This leads naturally to the following definition. A function $f$ defined on $[a, b]$ is said to be Henstock or Kurzweil-Henstock integrable if there is a real number $A$ for every $\varepsilon > 0$ there is $\delta(\xi) > 0$ such that for any $\delta$-fine division $D$ we have

$$|s(f, D) - A| < \varepsilon.$$

The earlier discussion shows that every Newton integrable function is Henstock integrable. Note that in the definition of the Henstock integral we use fewer divisions than in that of the McShane integral. Hence the Henstock integral includes the McShane and therefore the Lebesgue integrals. This is the constructive definition. Next, we shall give the descriptive definition and the definition using major and minor functions.

If $f$ is Henstock integrable on $[a, b]$, then its primitive $F$ is continuous but no longer absolutely continuous. The property that characterizes the primitive $F$ of a Henstock integrable function $f$ is $ACG^*$ or generalized absolute continuity (in the restricted sense). This is in a sense a countable extension of the concept of absolute continuity. Let $X \subset [a, b]$. A function $F$ is said to be $AC^*(X)$ if for every $\varepsilon > 0$ there is $\eta > 0$ such that for any finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$ with at least one endpoint $a_i$ or $b_i$ belonging to $X$ for each $i$ and satisfying

$$\sum_i |b_i - a_i| < \eta \quad \text{we have} \quad \sum_i |F(b_i) - F(a_i)| < \varepsilon.$$

The definition here is stated differently from the standard one in Saks [13]. But they are equivalent. A function $F$ is $ACG^*$ if $[a, b] = \bigcup_{i=1}^{\infty} X_i$ and $F$ is $AC^*(X_i)$ for each $i$.

**Theorem 5:** *If $f$ is Henstock integrable on $[a, b]$, then the primitive $F$ of $f$ is $ACG^*$.*

A descriptive definition of the Henstock integral is as follows and it is known as the Denjoy integral. A function $f$ defined on $[a, b]$ is said to be Denjoy integrable if there is an $ACG^*$ function $F$ such that $F'(x) = f(x)$ almost everywhere in $[a, b]$. It has been proved many times that the Henstock and Denjoy integrals are equivalent.

In the definitions of the Riemann and the Lebesgue integrals using major and minor functions, we impose the conditions piecewise linear and bounded variation respectively. If we drop all the conditions, we arrive at the definition for the Henstock integral.

**Theorem 6:** *A function f is Henstock integrable on [a, b] if and only if f is Perron integrable there, i.e., f has the major and minor functions such that*

$$\inf\{H(b) - H(a)\} = \sup\{G(b) - G(a)\}$$

*where the infimum is over all major functions H and the supremum is over all minor functions G.*

We remark that it does not matter whether we impose the additional condition that the major and minor functions are continuous. The two versions of the Perron integral are indeed equivalent. However if we impose the continuity, we may relax the inequalities to

$$\bar{D}G(x) \leq f(x) \leq \underline{D}\, H(x)$$

nearly everywhere, i.e. everywhere except perhaps for a countable number of points. If we want to relax the inequalities to hold almost everywhere, then we have to impose the $ACG^*$ property on the major and minor functions in order to be equivalent to the original definition.

So far the Henstock theory is still the simplest among all the existing integrals. It has been extended to $n$-dimensional and made abstract. A book on the abstract version of the Henstock integral or what Henstock calls a general theory of integration is in preparation by Henstock to be published by Oxford University Press in January 1991.

## §5. Recent advances

Recently, there has been active research on the subject of the Kurzweil-Henstock integral particularly the last five years. The theory has now been perfected, extended to higher dimensions and abstract version, and also applied to ordinary differential equations, trigonometric series, and functional analysis. Whether the research will remain active depends on how well the subject can penetrate into other branches of mathematics.

As a result of recent activities, a series of books have been published, including Chelidze and Djvarsheishvili [1], Henstock [3], Kurzweil [5], Lee [6], McShane [8], Muldowney [10], Ostaszewski [11], Schwabik, Turby and Vejvoda [14], Thomson [15]. Note that over half of them were published in and after 1984. There are comprehensive survey papers on the history and the development of classical integration theory by Bullen and by Henstock published in the Southeast Asian Bulletin of Mathematics. Also, many interesting research articles appeared in Real Analysis Exchange, a journal published in the United States, including papers by P. S. Bullen, T. S. Chew, G. Cross, C. S. Ding, R. Gordon, S. Leader, P. Y. Lee, B. L. Li, G. Q. Liu, S. P. Lu, K. M. Ostaszewski, W. F. Pfeffer, B. Thomson, D. F. Xu and others.

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